Linear and Combinatorial Optimization

- Repetition of duality
- The Knapsack problem
- The dual simplex method
- The revised simplex method
- Sensitivity analysis
The dual of an LP-problem

**Definition 4.1**

Let

\[
(P) \begin{cases}
\text{max } z = c^T x \\
Ax \leq b \\
x \geq 0
\end{cases}
\]

be an LP-problem in standard form (primal). The **dual** to the above problem is defined as

\[
(D) \begin{cases}
\text{min } v = b^T y \\
A^T y \geq c \\
y \geq 0
\end{cases}
\]

**Theorem 4.1**

*The dual of its dual is again the primal problem.*
## Primal-Dual Table

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>↔ min</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y \geq 0$</td>
</tr>
<tr>
<td>$Ax = b$</td>
<td>$y$ unc.</td>
</tr>
<tr>
<td>$Ax \geq b$</td>
<td>$y \leq 0$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$A^T y \geq c$</td>
</tr>
<tr>
<td>$x$ unc.</td>
<td>$A^T y = c$</td>
</tr>
<tr>
<td>$x \leq 0$</td>
<td>$A^T y \leq c$</td>
</tr>
</tbody>
</table>
Weak Duality Theorem

**Theorem 4.2 (Weak Duality)**

If \( x \) is a feasible solution to the primal problem and \( y \) is a feasible solution to the dual problem, then \( c^T x \leq b^T y \).

**Corollary 4.1**

(a) If the primal problem has a feasible solution, but the objective function is unbounded, then the dual has no feasible solution.

(b) If the dual problem has a feasible solution, but the objective function is unbounded, then the primal has no feasible solution.

It may happen that neither the primal nor the dual has a feasible solution!

**Theorem 4.3**

If \( x \) and \( y \) are feasible solutions to the primal and the dual, respectively, and if the objective function values are equal (\( c^T x = b^T y \)) then both \( x \) and \( y \) are optimal solution to their problems.
Complementary Slackness

Definition 4.2
Let $x$ denote a feasible solution to the primal problem and $y$ a feasible solution to the dual problem. The primal solution $x$ and the dual solution $y$ fulfill the complementary slackness condition (CS) if

$$
y^T (Ax - b) \leq 0 \quad x^T (A^T y - c) \leq 0
$$

Interpretation: $Ax \leq b$ active or $y = 0$.

Lemma 4.1
If $x$, $y$ fulfils CS then

$$c^T x = b^T y .$$

Theorem 4.4
If $x$ is optimal for $(P)$ and $y$ is optimal for $(D)$, then CS holds.
The Strong Duality Theorem

**Theorem 4.5**

*Strong duality* If $x$ is optimal for $(P)$ and $y$ is optimal for $(D)$, then

$$c^T x = b^T y.$$ 

**Theorem 4.6**

The complete duality theorem

(i) If $x$ is feasible and optimal for $(P)$ then there exists a $y$ that is feasible and optimal for $(D)$ and $c^T x = b^T y$.

(ii) If $(P)$ is unbounded then $(D)$ has no feasible solution.

(iii) If $(P)$ has no feasible solution then $(D)$ is either unbounded or has no feasible solution.
Using the dual problem and the Dual Simplex method

Idea 1: Apply Two-Phase-Simplex method to dual problem

Idea 2: Use dual problem when you have infeasible but optimal bs

A basic solution $B^{-1}b$ to the problem $\max c^T x, Ax \leq b, x \geq 0$, is feasible if $B^{-1}b \geq 0$ and optimal if $\hat{c}_N^T = c_N^T - c_B^T B^{-1} N \leq 0$.

Simplex method: from nonoptimal to optimal while feasible.

Dual Simplex method: from nonfeasible to feasible while optimal.

The solution $y = B^{-T} c_B$ becomes 'more' feasible.

1. Find basic solution such that $\hat{c}_N^T = c_N^T - c_B^T B^{-1} N \geq 0$.
2. Select departing basic variable (choose entry with $B^{-1}b$ most negative. If all are positive, you are done).
3. Select entering basic variable whose ratio between objective row and pivotal row is largest and negative.
4. Perform pivoting and go to 2.
Dual Simplex tableau method

\[ \text{max } c^T x, \ Ax = b, \ x \geq 0. \text{ Simplex tableau} \]

\[
\begin{pmatrix}
T \\ v^T \\ z
\end{pmatrix}
= 
\begin{pmatrix}
B^{-1} A \\ c_B^T B^{-1} A - c^T \\ c_B^T B^{-1} b
\end{pmatrix}
\]

where \( B \) and \( c_B^T \) consist of columns in \( A \) and \( c^T \), respectively, corresponding to basic variables.

There are several simple relations in the tableau.
If \( A \) contains the identity matrix, then one can find \( B^{-1} \) directly from the tableau.
If \( A \) contains the identity matrix and the corresponding entries of \( c^T \) are zero, then one can read off \( c_B^T B^{-1} = y^T \) directly.
Revised Simplex method

Idea: Don’t store the whole tableau and remove a lot of zeros and the identity matrix.

1. Selecting basic variables determines $B^{-1}$ and $c_B$.
2. Compute $v = c_B^TB^{-1}A - c^T$. Choose entering variable $k$ among most negative entries of $v$.
3. Compute $t = B^{-1}a_k$ and $x_B = B^{-1}b$. Form $\theta$-ratios for entries in $t$ which are positive. Select departing variable $q$ by lowest $\theta$-ratio.
4. Update $(B^{-1})_{new} = E(B^{-1})_{old}$, where

\[
E = \begin{pmatrix}
1 & \ldots & -t_1/t_q & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 1/t_q & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & -t_n/t_q & \ldots & 1
\end{pmatrix}
\]

5. Goto 2.
Sensitivity analysis

What happens when the inputs change slightly?

1. Change a coefficient in \( c \). The extreme points remain the same. One can increase the coefficients for nonbasic variables as much as the relative cost \( \hat{c}_N = c_N - N^T B^{-T} c_B \leq 0 \) without changing the optimal solution. It does not change the optimal objective value. The coefficient for a basic variable may also be modified a little without changing the optimal solution. How much is given by \( \hat{c}_N = c_N - N^T B^{-T} c_B \leq 0 \). Since \( z^* = c^T x^* \), the change in objective is \( \frac{dz}{dc_i} = x_i^* \).

2. Change a coefficient in \( b \). The extreme points move. One can change \( b \) as long as \( B^{-1} b \geq 0 \). Since \( z = c_B^T B^{-1} b = y^T b \), so \( \frac{dz}{db_i} = y_i \). The marginal profit depends on the corresponding dual variable’s value in the optimal solution, \( y \). \( c_B^T B^{-1} \) are sometimes called shadow prizes.
Repetition - Lecture 4

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- The dual simplex algorithm
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