Image Analysis - Motivation
Overview – Linear Algebra and FFT

1. Linear Algebra
   1. Vector space – ’A matrix is a vector’ What does this mean?
   2. Basis, coordinates
   3. Scalar product
   4. Projection onto a subspace
   5. Projection onto an affine ‘subspace’
   6. (Principal Component Analysis – Recipe)
   7. Change of basis

2. Fourier Transform
Vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$

The following linear spaces are well-known:

- $\mathbb{R}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{R}$

- $\mathbb{C}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{C}$
Basis

Definition
$e_1, \ldots, e_n \in \mathbb{R}^n$ is a basis in $\mathbb{R}^n$ if

- they are linearly independent
- they span $\mathbb{R}^n$.

Example (3D space)
$e_1, e_2, e_3 \in \mathbb{R}^3$ is a basis in $\mathbb{R}^3$ if they are not located in the same plane.
Canonical basis (normal basis)

Example (canonical basis)

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \]

is called the **canonical basis** in \( \mathbb{R}^n \).

\[ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 \ldots + x_n e_n . \]
Coordinates

Let \( e_1, e_2, \ldots, e_n \) be a basis. Then for every \( x \) there is a unique set of scalars \( \xi_i \) such that

\[
x = \sum_{i=1}^{n} \xi_i e_i.
\]

These scalars are called the **coordinates** for \( x \) in the basis \( e_1, e_2, \ldots, e_n \).
Scalar product

Definition
Let \( A \) be a (complex) matrix. Introduce
\[
A^* = (\bar{A})^T .
\]

Definition
Let \( x \) and \( y \) be two vectors in \( \mathbb{R}^n \) (\( \mathbb{C}^n \)). The scalar product of \( x \) and \( y \) is defined as
\[
x \cdot y = \sum \bar{x}_i y_i = x^* y .
\]
General Vector Space

• A ’General’ Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by ’numbers’ called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.

\[
\begin{align*}
(i) & \quad \bar{u} + \bar{v} = \bar{v} + \bar{u} & \text{(commutativity)} \\
(ii) & \quad (\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w}) & \text{(associativity)} \\
(iii) & \quad \bar{v} + \bar{0} = \bar{v} & \text{(zero existence)} \\
(iv) & \quad \bar{v} + (-\bar{v}) = \bar{0} & \text{(negative vector existence)} \\
(v) & \quad k(l\bar{v}) = (kl)\bar{v} & \text{(associativity)} \\
(vi) & \quad 1\bar{v} = \bar{v} & \text{(multiplicative one)} \\
(vii) & \quad 0\bar{v} = \bar{0} & \text{(multiplicative zero)} \\
(viii) & \quad k\bar{0} = \bar{0} & \text{(multiplicative zero vector)} \\
(ix) & \quad k(\bar{u} + \bar{v}) = k\bar{u} + k\bar{v} & \text{(distributivity 1)} \\
(x) & \quad (k + l)\bar{v} = k\bar{v} + l\bar{v} & \text{(distributivity 2)}
\end{align*}
\]
General Vector Space

- A ’General’ Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by ’numbers’ called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.

- There are many examples of such vectors spaces. The vectors can for example be
  - Geometrical vectors in three dimensions
  - N-tuples of real numbers
  - Functions
  - Polynomials
  - Matrices
  - Tensors
Example - polynomials

- Vectors - Polynomials of degree 2
- Scalars – Real numbers

Example 3.2.1. Polynomials in one variable of degree 2 is a vector space. One possible basis is

\[ \bar{e}_1(x) = 1, \quad \bar{e}_2(x) = x, \quad \bar{e}_3(x) = x^2. \]

The polynomial \( \bar{u}(x) = 5x^2 + 3x - 2 \) has coordinates \( u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \) since

\[
\bar{u} = \underbrace{u_1}_{-2} \bar{e}_1 + \underbrace{u_2}_{3} \bar{e}_2 + \underbrace{u_3}_{5} \bar{e}_3 = 5x^2 + 3x - 2.
\]

The dimension of the vector space is 3.
Example - matrices

- Vectors – Matrices of size 2x2
- Scalars – Real numbers

Example 3.2.2. Matrices of size $2 \times 2$ is a vector space. One possible basis is

$$\tilde{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{e}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tilde{e}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$\bar{u} = \begin{pmatrix} 1 \\ 7 \\ 3 \\ 2 \end{pmatrix}$$

has coordinates $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \\ 2 \end{pmatrix}$, since

$$\bar{u} = u_1 \tilde{e}_1 + u_2 \tilde{e}_2 + u_3 \tilde{e}_3 + u_4 \tilde{e}_4 = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

The dimension of the vector space is 4.
## Image matrix

\[ f = \begin{bmatrix} f(1, 1) & f(1, 2) & \cdots & f(1, N) \\ f(2, 1) & f(2, 2) & \cdots & f(2, N) \\ \vdots & \vdots & \ddots & \vdots \\ f(M, 1) & f(M, 2) & \cdots & f(M, N) \end{bmatrix} \]

\[ f(j, \cdot) = \begin{bmatrix} f(j, 1) & f(j, 2) & \cdots & f(j, N) \end{bmatrix}, \]

\[ f(\cdot, k) = \begin{bmatrix} f(1, k) \\ f(2, k) \\ \vdots \\ f(M, k) \end{bmatrix}. \]
Column stacking

\[ \tilde{f} = \begin{bmatrix} f(\cdot, 1) \\ f(\cdot, 2) \\ \vdots \\ f(\cdot, N) \end{bmatrix} \]

\[ \widetilde{f + g} = \tilde{f} + \tilde{g}, \quad \lambda \tilde{f} = \lambda \tilde{f} \]
Set of images is a vector space

- Images are a vector space (with scalar product)
  - Addition
  - Multiplication by scalar
- Two ways to think of 'images' as vectors (both are the same)
  - 1. Column stacking
    - Use column stacking to convert to 'old school' vector $\mathbb{R}^n$
    - Use linear algebra as usual
    - Convert back to matrix form when needed
  - 2. Image basis
    - Choose a basis (any basis).
    - Through the use of coordinates, obtain vector representation
    - Use linear algebra as usual
    - Convert back when needed
Overview – Linear Algebra and FFT

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2. Fourier Transform
Canonical basis

\[ \chi(i, j) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}, \]

with the 1 at position \((i, j)\).

Using this canonical basis we can write

\[ f = \sum_{i,j} f(i, j) \chi(i, j). \]

Idea for image transform:
Choose another basis that is more suitable in some sense.
Image matrices can thus be seen as vectors in a linear space.
Scalar product of images

Definition
The scalar product of two matrices (images) is defined as

\[ f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j)g(i,j). \]

\( x, y \in \mathbb{R}(\mathbb{C}) \) are orthogonal if \( x \cdot y = 0 \). This is often written

\[ x \perp y \iff x \cdot y = 0. \]

The length or the norm of \( x \) is defined as

\[ \| x \| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j)f(i,j)}. \]
Scalar product and norm

Example 3.2.1 (Scalar product and norm). Let
\[
f = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}
\]
and
\[
g = \begin{pmatrix} 4 & 2 \\ -1 & -3 \end{pmatrix}.
\]

What is the scalar product \( f \cdot g \)? What is the norm \( ||f|| \)?

\[
f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} f(i, j) g(i, j).
\]

\[
||f|| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} f(i, j) f(i, j)}.
\]
Orthonormal basis

Definition
\( \{ e_1, \ldots, e_n \} \) is an **orthonormal (ON-) basis** in \( \mathbb{R}^n (\mathbb{C}^n) \) if

\[
e_i \cdot e_j = \begin{cases} 
0 & i \neq j \\
1 & i = j 
\end{cases}
\]
Theorem
Assume that \( \{e_1, \ldots, e_n\} \) is orthonormal (ON) basis and

\[
x = \sum_{i=1}^{n} \xi_i e_i.
\]

Then

\[
\xi_i = e_i \cdot x = e_i^* x, \quad \|x\|^2 = \sum_{i=1}^{n} |\xi_i|^2
\]
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2. Fourier Transform
Orthogonal projection

Definition
Let \( \{a_1, \ldots, a_k\} \in \mathbb{R}^n, k \leq n \), span a linear subspace, \( \pi \), in \( \mathbb{R}^n \), i.e.:

\[
\pi = \{w | w = \sum_{i=1}^{k} x_i a_i, x_i \in \mathbb{R}\}.
\]

The **orthogonal projection** of \( u \in \mathbb{R}^n \) on \( \pi \) is a linear mapping \( P \), such that \( u_{\pi} = Pu \) and defined by

\[
\min_{w \in \pi} \|u - w\| = \|u - u_{\pi}\|.
\]

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Image Analysis - Lecture 2
Orthogonal projection

The orthogonal projection is characterized by

1. $u_\pi \in \pi$
2. $u - u_\pi \perp w$ for every $w \in \pi$
Let \( a \in \pi \) and \( b \in \pi \) be two solutions to the minimisation problem. Set

\[
f(t) = \|u - ta - (1 - t)b\|^2 = \ldots
\]

\[
= \|u - b\|^2 + t^2\|a - b\|^2 - 2t(a - b) \cdot (u - b), \quad t \in \mathbb{R}.
\]

This is a second degree polynomial with minimum in \( t = 0 \) and \( t = 1 \) \( \Rightarrow \) \( f(t) \) is a constant function and thus \( \Rightarrow a = b \).
Let \( f(t) = \|u - u_\pi + ta\|^2 \), where \( a \in \pi \). It follows that \( f'(0) = 2(u - u_\pi) \cdot a = 0 \), i.e. \((u - u_\pi) \perp a\).

Conversely: Assume \( w \in \pi \). The property that \((u - u_\pi) \perp a\) for every \( a \in \pi \) gives that

\[
\|u - w\|^2 = \|u - u_\pi + u_\pi - w\|^2 = \\
\|u - u_\pi\|^2 + \|u_\pi - w\|^2 \geq \|u - u_\pi\|^2,
\]

i.e. \( u_\pi \) solves the minimization problem.
Let $A = [a_1 \ldots a_k]$ be a $n \times k$ matrix and

$$\pi = \{ w | w = Ax, \ x_i \in \mathbb{R}^n \}$$

**Lemma**

If $\{a_1, \ldots, a_k\}$ are linearly independent $\mathbb{R}^n$ then $A^*A$ is invertible.

**Proof:** Do it on your own. (Use SVD if you are familiar with it.)
Theorem

If the columns of $A$ are linearly independent, then the projection of $u$ on $\pi$ is given by

$$u_\pi = x_1 a_1 + \ldots + x_k a_k, \quad x = (A^* A)^{-1} A^* u.$$  

Proof: Use the characterization of the projection (above).

$$a_i^* (u - u_\pi) = 0 \quad \Rightarrow$$

$$A^* (u - Ax) = 0 \quad \Rightarrow$$

$$A^* u = A^* Ax \quad \Rightarrow \quad x = (A^* A)^{-1} A^* u$$
Definition

$A^+ = (A^* A)^{-1} A^*$ is called the **pseudo-inverse** of $A$. 

Observe that if $A$ is quadratic and invertible then $A^+ = A^{-1}$.

**Theorem**

If $\{a_1, \ldots, a_k\}$ are orthonormal, then the projection of $u$ on $\pi$ is given by

$$u_\pi = y_1 a_1 + \ldots + y_k a_k, \quad y_i = a_i^* u.$$

**Proof:** This follows from $A^* A = I$. 

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Illustration
Orthogonal projection

What is the orthogonal projection of \( f \)

\[
f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}
\]

onto the space spanned by \((e_1, e_2, e_3)\)

\[
e_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \quad e_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}
\]
Orthogonal projection

\[ f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix} \]

Since \((e_1, e_2, e_3)\) is orthonormal the coordinates are
\[ x_1 = f \cdot e_1 = 14, \quad x_2 = f \cdot e_2 = -15/\sqrt{6}, \quad x_3 = f \cdot e_3 = -4/\sqrt{6}. \]

The orthogonal projection is then
\[ \hat{f} = 14e_1 - 15/\sqrt{6}e_2 - 4/\sqrt{6}e_3 \]

\[ f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}, \quad \hat{f} = \begin{pmatrix} 1.5 & 2^{1/6} & 2^{5/6} \\ 4 & 4^{1/3} & 5^{1/3} \\ 6.5 & 7^{1/6} & 7^{1/6} \end{pmatrix}, \]
What is the orthogonal projection of $f$ onto the space spanned by $(e_1, e_2, e_3)$?
Since \((e_1, e_2, e_3)\) is orthonormal, the coordinates are
\[ x_1 = f \cdot e_1 = -2457, \quad x_2 = f \cdot e_2 = 303, \quad x_3 = f \cdot e_3 = -603. \]
The orthogonal projection is then
\[ \hat{f} = -2457e_1 + 303e_2 - 603e_3. \]
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2. Fourier Transform
Projection onto affine subspace

- Previously projection onto linear subspace
  \[ \pi = \{ w \mid w = \sum_{i=1}^{n} x_i a_i = Ax \quad \text{where} \quad x_i \in \mathbb{C} \text{ (or} \mathbb{R}) \} \]

- A linear subspace always contains the zero vector

- How about planes or 'subspaces' that are shifted away from the origin. Such sets are called affine spaces.
  \[ \pi = \{ w \mid w = m + \sum_{i=1}^{n} x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C} \text{ (or} \mathbb{R}) \} \]

- An affine subspace is typically not a linear space
Projection onto affine subspace

- An affine subspace, defined by \( m, a_1, \ldots, a_k \).

\[
\pi = \{ w \mid w = m + \sum_{i=1}^{n} x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C} \text{ (or } \mathbb{R}) \}.
\]

- Projection of \( u \) onto the affine subspace
  - Subtract \( m \), i.e. form \( v = u - m \).
  - Project \( v \) onto the space spanned by \( a_1, \ldots, a_k \), i.e. \( v_\pi = A^+ v \).
  - Add \( m \), i.e. form \( u_\pi = v_\pi + m \).
Overview –
Linear Algebra and FFT

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2. Fourier Transform
PCA - Principal Component Analysis

- Orthogonal projection – project an image $u$ on
  - subspace spanned by $a_1, \ldots, a_k$.
  - or affine subspace defined by $m$, $a_1, \ldots, a_k$.

- How do we find a good subspace?
- Given lots of vectors $x_1, \ldots, x_N$. Find a suitable affine subspace so that the orthogonal projections $y_i$ of $x_i$ are as close to $x_i$ as possible

$$e(\pi) = \sum_{i=1}^{N} ||y_i(\pi) - x_i||^2.$$
PCA - Principal Component Analysis

1. Calculate the mean $m = \frac{1}{N} \sum_{i=1}^{N} x_i$.

2. Subtract the mean from all examples $z_i = x_i - m$.

3. Place all of the resulting vectors as columns of a matrix, $M = (z_1 \ldots z_N)$.

4. Factorize $M$ using the singular value decomposition $M = USV^T$.

5. Use the first $k$ columns of $U$ as the basis of the subspace, i.e. $a_i = u_i$, with $U = (u_1 \ldots u_m)$.

$$\pi = \{ w \mid w = m + \sum_{1}^{n} x_i a_i = Ax + m \text{ where } x_i \in \mathbb{C} \text{ (or } \mathbb{R} \} \}.$$

$$e(\pi) = \sum_{i=1}^{N} \| y_i(\pi) - x_i \|^2.$$
PCA – “Training”
Given examples, find subspace
PCA - Principal Component Analysis
PCA - Principal Component Analysis

\[ w = m + \sum_{1}^{n} x_i a_i \]
PCA - Principal Component Analysis

Given vectors $\mathbf{A}$, $\mathbf{m}$, $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{w}$, the projection of $\mathbf{m}$ onto the affine subspace is given by $w = m + \sum_{i=1}^{n} x_i a_i$.
PCA - Principal Component Analysis

\[ w = m + \sum_{1}^{n} x_i a_i \]
PCA - Principal Component Analysis

\[ w = m + \sum_{i=1}^{n} x_i a_i \]
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2. Fourier Transform
Fourier Transform

\[ F(u, v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x, y) e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)} \]

- Can be viewed as a change of basis
- Image f \rightarrow Fourier Transform F (and back)
- Has strong connections with convolutions
- (next lecture)
- Useful for image compression
- Useful for image understanding
- Basically a great tool
Fourier Transform

- Definition, is a change of basis, what does is mean
- Detour (for increased understanding
- Ordinary Fourier Transform (from previous courses)
- Examples
- Properties
- Discrete Fourier Transform – 1D
Image basis example (Walsh)

\[ f = \begin{bmatrix} 9 & -1 \\ 5 & 7 \end{bmatrix} \]

\[ \Phi_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} / 2 \quad \Phi_{12} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} / 2 \]

\[ \Phi_{21} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} / 2 \quad \Phi_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} / 2 \]

\[ x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda, \mu} f(\lambda, \mu) \Phi_{ij}(\lambda, \mu) \]

\[ f = x_{11} \Phi_{11} + x_{21} \Phi_{21} + x_{12} \Phi_{12} + x_{22} \Phi_{22} \]

\[ x = \begin{bmatrix} 10 & 4 \\ -2 & 6 \end{bmatrix} \]

- Image \( f \rightarrow \) Fourier Transform \( x \) (and back)
Fourier transform as change of image basis

\[ x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda, \mu} f(\lambda, \mu) \Phi_{ij}(\lambda, \mu) \]

\[ f = x_{11} \Phi_{11} + x_{21} \Phi_{21} + x_{12} \Phi_{12} + x_{22} \Phi_{22} \]

\[ F(u, v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x, y) e^{-i 2\pi ((u-1)(x-1)/M + (v-1)(y-1)/N)} \]

\[ f(x, y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u, v) e^{i 2\pi ((u-1)(x-1)/M + (v-1)(y-1)/N)} \]
Compare with ordinary Fourier Transform

Definition

Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$. The Fourier transform of $f$ is defined as

$$(\mathcal{F} f)(u) = F(u) = \int_{-\infty}^{+\infty} e^{-i2\pi xu} f(x) dx.$$  

Theorem

Under the right assumptions on $f$, the following inversion formula

$$f(x) = \int_{-\infty}^{+\infty} e^{i2\pi ux} F(u) du$$

holds.
Examples

\[ \delta(x) \mapsto 1(u) \]
\[ \text{rect}(x) \mapsto 2 \frac{\sin(2\pi u)}{2\pi u} = 2 \text{sinc}(2\pi u) \]
Examples

- Review of Linear algebra
- Fourier transform
  - Continuous Fourier transform
  - Discrete Fourier Transform (DFT, FFT)
- Two-dimensional Fourier Transform

Illustrations

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Image Analysis - Lecture 2
Examples

\[ c_1 f_1(x) + c_2 f_2(x) \mapsto c_1 F_1(u) + c_2 F_2(u) \]  (linearity)

\[ f(\lambda x) \mapsto \frac{1}{|\lambda|} F\left(\frac{u}{\lambda}\right) \]  (scaling)

\[ f(x - a) \mapsto e^{-i2\pi u a} F(u) \]  (translation)

\[ e^{-i2\pi x a} f(x) \mapsto F(u + a) \]  (modulation)

\[ \overline{f(x)} \mapsto F(-u) \]  (conjugation)

\[ \frac{df}{dx} \mapsto 2\pi i u F(u) \]  (differentiation I)

\[ -2\pi i x f(x) \mapsto \frac{dF}{du} \]  (differentiation II)

Example: \[ \delta(x - 1) \mapsto e^{-i2\pi u} \]
The DFT can then be written as a matrix multiplication. Let 

\[ f = \begin{bmatrix} f(1) \\ \vdots \\ f(N) \end{bmatrix} \]

Definition 3.3.2.

The equation can be seen as a discretized version of the continuous version of the Fourier transform. For a

is one of the

is important.

\[
\begin{align*}
F(u) &= \sum_{x=1}^{N} f(x) \exp[-i2\pi(u - 1)(x - 1)/N] \\
\omega_N &= \exp(-i2\pi/N)
\end{align*}
\]

We introduce the complex constant
Discrete Fourier Transform - 1D

Definition
The Fourier Matrix $\mathcal{F}_N$ is given by

$$\mathcal{F}_N = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega_N & \omega_N^2 & \ldots & \omega_N^{N-1} \\
1 & \omega_N^2 & \omega_N^4 & \ldots & \omega_N^{2(N-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \ldots & \omega_N^{(N-1)(N-1)}
\end{pmatrix}.$$ 

$$f \quad \mapsto \quad F = \mathcal{F}_N f.$$
Discrete Fourier Transform - 1D

**Theorem 3.3.1.** For the Fourier matrix the following holds,

$$\mathcal{F} \mathcal{F} = NI.$$  

From this we obtain $\mathcal{F}^{-1} = \frac{1}{N} \mathcal{F}$ The inverse Fourier transform is thus

$$f = \mathcal{F} F \iff f(x) = \frac{1}{N} \sum_{u=1}^{N} F(u) \omega_N^{(x-1)(u-1)}, \quad x = 1, \ldots, N.$$
Discrete Fourier Transform - 1D

- Important: DFT assumes that signals are periodic!
- Think of the signal as wrapped periodically

- Fourier transform is complex.
- Plot absolute value and phase

- Low frequencies in the edges/corners.
- Ordinary images typically have large values for low frequencies.
Discrete Fourier Transform - 2D

\[
F(u, v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x, y) e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}
\]

\[
f(x, y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u, v) e^{i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}
\]
Discrete Fourier Transform - 2D

Let the matrix $F$ represent the Fourier transform of the image $f(x, y)$:

$$F = \mathcal{F}_M f \mathcal{F}_N$$

or

$$F = \mathcal{F}_M (\mathcal{F}_N f^T)^T .$$

i.e. the DFT in two dimensions can be calculated by repeated use of the one-dimensional DFT, first for the rows, then for the columns.
Discrete Fourier Transform - 2D Example

Low frequencies in the edges/corners
Fourier transform is complex. Plot absolute value and phase
Discrete Fourier Transform - 2D Example – Periodic expansion
Discrete Fourier Transform - 2D Example – Periodic expansion

\[(1-M \times 1-N)\]
Discrete Fourier Transform - 2D Example – Periodic expansion

((-M/2 - M/2) x (-N/2 - N/2))
Discrete Fourier Transform - 2D Example – Periodic expansion

\[ (-M/2 - M/2 \times -N/2 - N/2) \]

fftshift
• Usually, the gray-levels of the Fourier Transform images are scaled using $c \log(1 + |F(u, v)|)$.
• The middle of the Fourier image (after fftshift) corresponds to low frequencies.
• Outside the middle high components in $F$ corresponds to higher frequencies and the direction corresponds to "edges" in the images with opposite orientation.
What does the original image look like if this is the Fourier transform?

Left: Magnitude, Right: Phase
Review of Linear algebra
Fourier transform
Continuous Fourier transform
Discrete Fourier Transform (DFT, FFT)
Two-dimensional Fourier Transform

Example

Kalle Åström
Image Analysis - Lecture 2
Fourier transform

• Image
Fourier transform

- Image
- \( \text{abs}(\text{fft2}(I)) \)
Fourier transform

- Image
- \( \log(\text{abs}(\text{fft2}(I))) \)
Edge effects

- Image
- $\log(|\text{fft2}(I)|)$
Fourier transform

• Image
Fourier transform

• Image

• Fourier transform
Fourier transform

• Image
Fourier transform

- Image
- Fourier transform
Review

- Linear algebra
  - The space of images is a linear vector space
  - Images are ‘vectors’ – in the sense that they are elements of a linear vectors space
  - Can be confusing. Can a matrix be a vector???
- Useful tools
  - Change of basis
  - Projection onto a subspace, onto affine subspace
  - PCA
  - Fourier Transform

- Read lecture notes
- Experiment with matlab demo scripts
- Continue with assignment 1
Master’s Thesis Suggestion of the day

- Recognise and label parts of 3D models.
- Eye, mouth, hair, nose, …