Chapter V. Distributions in Product Spaces

Summary

We were not able to define the product of arbitrary distributions in Chapter III. However, as we shall now see this can always be done when they depend on different sets of variables. Thus to arbitrary distributions $u_j \in \mathscr{D}'(X_j)$, X_j open in $\mathbb{R}^{n_j}$ $(j=1,2)$, we define in Section 5.1 a product $u_1 \otimes u_2 \in \mathscr{D}'(X_1 \times X_2)$ in $X_1 \times X_2 \subset \mathbb{R}^{n_1+n_2}$. In case u_j are functions this is the function $X_1 \times X_2 \ni (x_1, x_2) \to u_1(x_1)u_2(x_2)$.

On the other hand, a function $K \in C(X_1 \times X_2)$ can be viewed as the kernel of an integral operator $\mathscr{K}$,

$$(\mathscr{K} u)(x_1) = \int K(x_1, x_2) u(x_2) dx_2,$$

mapping $C_0(X_2)$ to $C(X_1)$ say. It is not easy to characterize the operators having such a kernel. However, the analogue in the theory of distributions is very satisfactory. It is called the Schwartz kernel theorem and states that the distributions $K \in \mathscr{D}'(X_1 \times X_2)$ can be identified with the continuous linear maps $\mathscr{K}$ from $C_0^\infty(X_2)$ to $\mathscr{D}'(X_1)$ which they define. This will be proved in Section 5.2. We shall return to this topic in Section 8.2. A rather precise classification of singularities will then allow us to discuss the regularity of $\mathscr{K} u$ and its definition when u is not smooth.

5.1. Tensor Products

If X_j is an open set in $\mathbb{R}^{n_j}$, $j=1,2$, and if $u_j \in C(X_j)$, then the function $u_1 \otimes u_2$ in $X_1 \times X_2 \subset \mathbb{R}^{n_1+n_2}$ defined by

$$(u_1 \otimes u_2)(x_1, x_2) = u_1(x_1) u_2(x_2), \qquad x_j \in X_j,$$

is called the direct (or tensor) product of u_1 and u_2. To extend the definition to distributions we observe that $u_1 \otimes u_2 \in C(X_1 \times X_2)$ and

that

$$\iint (u_1 \otimes u_2)(\phi_1 \otimes \phi_2)\,dx_1\,dx_2 = \int u_1 \phi\,dx_1 \int u_2 \phi_2\,dx_2, \qquad \phi_j \in C_0^\infty(X_j).$$

Theorem 5.1.1. *If $u_j \in \mathscr{D}'(X_j)$, $j=1,2$, then there is a unique distribution $u \in \mathscr{D}'(X_1 \times X_2)$ such that*

$$u(\phi_1 \otimes \phi_2) = u_1(\phi_1)\,u_2(\phi_2), \qquad \phi_j \in C_0^\infty(X_j). \tag{5.1.1}$$

We have

$$u(\phi) = u_1[u_2(\phi(x_1,x_2))] = u_2[u_1(\phi(x_1,x_2))], \qquad \phi \in C_0^\infty(X_1 \times X_2), \tag{5.1.2}$$

where u_j acts on the following function as a function of x_j only. If $u_j \in \mathscr{E}'$, $j=1,2$, the same formula is valid for $\phi \in C^\infty$. The distribution u is called the tensor product and one writes $u = u_1 \otimes u_2$.

Proof. a) Uniqueness. We must show that if $u \in \mathscr{D}'(X_1 \times X_2)$ and if

$$u(\phi_1 \otimes \phi_2) = 0 \qquad \text{for } \phi_j \in C_0^\infty(X_j),$$

then $u=0$. To do so we take $\psi_j \in C_0^\infty(\mathbb{R}^{n_j})$ with $\psi_j \geqq 0$, $\int \psi_j\,dx_j = 1$, and $|x_j| \leqq 1$ if $x_j \in \operatorname{supp}\psi_j$. With

$$\Psi_\varepsilon(x_1,x_2) = \varepsilon^{-n_1-n_2}\psi_1(x_1/\varepsilon)\,\psi_2(x_2/\varepsilon)$$

we know that $u * \Psi_\varepsilon \to u$ in $\mathscr{D}'(Y)$ if $Y \Subset X_1 \times X_2$ (Theorem 4.1.4). However, $u * \Psi_\varepsilon = 0$ in Y for small ε since $\Psi_\varepsilon(x_1-y_1, x_2-y_2)$ is the product of a function of y_1 and one of y_2. Hence $u=0$ in Y and therefore in X.

b) Existence of u and (5.1.2). Let K_j be a compact subset of X_j. Then

$$|u_j(\phi_j)| \leqq C_j \sum_{|\alpha| \leqq k_j} \sup|\partial^\alpha \phi_j|, \qquad \phi_j \in C_0^\infty(K_j).$$

If $\phi \in C_0^\infty(K_1 \times K_2)$ then

$$I_\phi(x_1) = u_2(\phi(x_1,.))$$

is in $C_0^\infty(K_1)$ by Theorem 2.1.3, and

$$\partial_{x_1}^\alpha I_\phi(x_1) = u_2(\partial_{x_1}^\alpha \phi(x_1,.)).$$

Hence

$$\sup|\partial_{x_1}^\alpha I_\phi(x_1)| \leqq C_2 \sum_{|\beta| \leqq k_2} \sup|\partial_{x_1}^\alpha \partial_{x_2}^\beta \phi(x_1,x_2)|$$

so $u_1(I_\phi)$ is defined and

$$|u_1(I_\phi)| \leqq C_1 C_2 \sum_{|\alpha_j| \leqq k_j} \sup|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\phi(x_1,x_2)|.$$

Writing $u(\phi)=u_1(I_\phi)$ we obtain a distribution satisfying (5.1.1) and the first part of (5.1.2). In the same way we obtain a distribution satisfying (5.1.1) and with the second property in (5.1.2). By the uniqueness both conditions (5.1.2) must therefore be valid. The remaining statement follows in the same way; note that

$$\operatorname{supp} u_1 \otimes u_2 = \operatorname{supp} u_1 \times \operatorname{supp} u_2. \tag{5.1.3}$$

Example 5.1.2. If δ_{a_j} is the Dirac measure at $a_j \in X_j$, then $\delta_{a_1} \otimes \delta_{a_2}$ is the Dirac measure δ_a at $a=(a_1, a_2) \in X_1 \times X_2$. Theorem 2.3.5 can now be stated as follows: If $u \in \mathscr{D}'(X_1 \times X_2)$ is of order k and if $\operatorname{supp} u \subset X_1 \times \{a_2\}$, $a_2 \in X_2$, then

$$u = \sum_{|\alpha| \leq k} u_\alpha \otimes \partial^\alpha \delta_{a_2}$$

where $u_\alpha \in \mathscr{D}'^{k-|\alpha|}(X_1)$ and α is a multi-index corresponding to the X_2 variables.

The direct product allows us to justify the definition (1.3.1)′ of the convolution in general. In fact, if $u_1, u_2 \in \mathscr{D}'(\mathbb{R}^n)$ and either one has compact support, then

$$(u_1 \otimes u_2)(\phi(x_1+x_2)) = (u_1 * u_2)(\phi), \qquad \phi \in C_0^\infty(\mathbb{R}^n), \tag{5.1.4}$$

for if $\check{\phi}(x)=\phi(-x)$ we have

$$u_2(\phi(x_1+x_2)) = (u_2 * \check{\phi})(-x_1)$$

so the left-hand side is $u_1 * (u_2 * \check{\phi})(0) = (u_1 * u_2) * \check{\phi}(0)$. (5.1.4) could also have been taken as definition of convolution. However, it was convenient to have convolution available in the proof of Theorem 5.1.1.

5.2. The Kernel Theorem

Every function $K \in C(X_1 \times X_2)$ defines an integral operator $\mathscr{K}$ from $C_0(X_2)$ to $C(X_1)$ by the formula

$$(\mathscr{K}\phi)(x_1) = \int K(x_1, x_2)\, \phi(x_2)\, dx_2, \qquad \phi \in C_0(X_2),\ x_1 \in X_1.$$

We shall now show that the definition can be extended to arbitrary distributions K if ϕ is restricted to C_0^∞ and $\mathscr{K}\phi$ is allowed to be a distribution. To do so we start from the observation that when $K \in C(X_1 \times X_2)$

$$\langle \mathscr{K}\phi, \psi \rangle = K(\psi \otimes \phi); \qquad \psi \in C_0^\infty(X_1),\ \phi \in C_0^\infty(X_2). \tag{5.2.1}$$

Theorem 5.2.1 (The Schwartz kernel theorem). *Every* $K \in \mathscr{D}'(X_1 \times X_2)$ *defines according to* (5.2.1) *a linear map* $\mathscr{K}$ *from* $C_0^\infty(X_2)$ *to* $\mathscr{D}'(X_1)$

which is continuous in the sense that $\mathscr{K}\phi_j \to 0$ in $\mathscr{D}'(X_1)$ if $\phi_j \to 0$ in $C_0^\infty(X_2)$. Conversely, to every such linear map $\mathscr{K}$ there is one and only one distribution K such that (5.2.1) *is valid. One calls K the kernel of $\mathscr{K}$.*

Proof. If $K \in \mathscr{D}'(X_1 \times X_2)$ then (5.2.1) defines a distribution $\mathscr{K}\phi$ since $\psi \to K(\psi \otimes \phi)$ is continuous; $\mathscr{K}$ is continuous since $\phi \to K(\psi \otimes \phi)$ is continuous. To prove the converse we first note that the uniqueness is identical to the uniqueness in Theorem 5.1.1. To prove the existence we observe that for any compact sets $K_j \subset X_j$ there are constants C, N_j such that

$$|\langle \mathscr{K}\phi, \psi \rangle| \leqq C \sum_{|\alpha| \leqq N_1} \sup |\partial^\alpha \psi| \sum_{|\beta| \leqq N_2} \sup |\partial^\beta \phi|; \tag{5.2.2}$$

$$\psi \in C_0^\infty(K_1), \quad \phi \in C_0^\infty(K_2).$$

In fact, by hypothesis the bilinear form

$$C_0^\infty(K_1) \times C_0^\infty(K_2) \ni (\psi, \phi) \to \langle \mathscr{K}\phi, \psi \rangle$$

is continuous with respect to ϕ (resp. ψ) for fixed ψ (resp. ϕ), and every separately continuous bilinear form in a product of Fréchet spaces is continuous.

Let $Y_j \Subset X_j$, choose the compact sets K_j as neighborhoods of $\bar{Y}_j$ and set for $(x_1, x_2) \in Y_1 \times Y_2$ and small $\varepsilon > 0$

$$K_\varepsilon(x_1, x_2) = \varepsilon^{-n_1 - n_2} \langle \mathscr{K}\psi_2((x_2 - .)/\varepsilon), \psi_1((x_1 - .)/\varepsilon) \rangle \tag{5.2.3}$$

where ψ_j are chosen as in the proof of Theorem 5.1.1. Note that if we already knew that there is a distribution K satisfying (5.2.1) then K_ε would be $K * \Psi_\varepsilon$ and therefore converge to K as $\varepsilon \to 0$. Our program is now to show that K_ε does have a limit in $\mathscr{D}'(Y_1 \times Y_2)$ when $\varepsilon \to 0$ and then to show that (5.2.1) is fulfilled for the limit.

(5.2.3) is well defined when ε is smaller than the distance from Y_j to $\complement K_j$, and by (5.2.2) we have with $\mu = N_1 + N_2 + n_1 + n_2$

$$|K_\varepsilon(x_1, x_2)| \leqq C \varepsilon^{-\mu} \quad \text{if } x_j \in Y_j,\ j = 1, 2. \tag{5.2.4}$$

We shall prove that K_ε has a limit in $\mathscr{D}'^{\mu+1}(Y_1 \times Y_2)$ as $\varepsilon \to 0$ by using an argument which is very close to the proof of Theorem 3.1.11. Note that if $\psi \in C^\infty(\mathbb{R}^n)$ then

$$\frac{\partial}{\partial \varepsilon}(\varepsilon^{-n}\psi(x/\varepsilon)) = \sum \frac{\partial}{\partial x_j}(\varepsilon^{-n}\psi_j(x/\varepsilon)), \qquad \psi_j(x) = -x_j \psi(x). \tag{5.2.5}$$

In fact, by the homogeneity

$$\varepsilon \frac{\partial}{\partial \varepsilon}(\varepsilon^{-n}\psi(x/\varepsilon)) + \sum x_j \frac{\partial}{\partial x_j}(\varepsilon^{-n}\psi(x/\varepsilon)) = -n\varepsilon^{-n}\psi(x/\varepsilon)$$

which implies (5.2.5). Now it follows from the continuity (5.2.2) that we may differentiate with respect to ε or x_j in (5.2.3), and by (5.2.5) this gives

$$\partial K_\varepsilon(x_1, x_2)/\partial\varepsilon = \sum_\nu \partial L^\nu_\varepsilon(x_1, x_2)/\partial x_\nu$$

where x_ν runs over all coordinates of (x_1, x_2). Here L^ν_ε is defined by replacing ψ_1 or ψ_2 by the product with $-x_\nu$, so (5.2.4) is valid for L^ν_ε. Repeating this process we conclude that

$$K^{(j)}_\varepsilon(x_1, x_2) = \partial^j K_\varepsilon(x_1, x_2)/\partial\varepsilon^j$$

is a sum of derivatives of order j of functions having a bound of the form (5.2.4), so $\varepsilon^\mu K^{(j)}_\varepsilon$ is bounded in $\mathscr{D}'^j(Y_1 \times Y_2)$ for every j. With a fixed small δ and $\varepsilon \to 0$ we now use Taylor's formula

$$K_\varepsilon = \sum_0^\mu (\varepsilon-\delta)^j K^{(j)}_\delta/j! + (\varepsilon-\delta)^{\mu+1} \int_0^1 K^{(\mu+1)}_{\delta+t(\varepsilon-\delta)}(1-t)^\mu/\mu!\, dt.$$

Since

$$(1-t)^\mu/(\delta + t(\varepsilon-\delta))^\mu \leqq \delta^{-\mu}$$

it follows for $\Phi \in C^{\mu+1}_0(Y_1 \times Y_2)$ that when $\varepsilon \to 0$

$$\begin{aligned}\langle K_\varepsilon, \Phi\rangle &\to \langle K_0, \Phi\rangle \\ &= \sum_0^\mu (-\delta)^j \langle K^{(j)}_\delta/j!, \Phi\rangle + (-\delta)^{\mu+1} \int_0^1 \langle K^{(\mu+1)}_{\delta(1-t)}, \Phi\rangle (1-t)^\mu/\mu!\, dt,\end{aligned}$$

where $K_0 \in \mathscr{D}'^{\mu+1}(Y_1 \times Y_2)$.

Let $\phi_j \in C^\infty_0(Y_j)$ and form

$$\langle K_\varepsilon, \phi_1 \otimes \phi_2\rangle = \iint K_\varepsilon(x_1, x_2)\phi_1(x_1)\phi_2(x_2)\, dx_1\, dx_2.$$

With the notation $\check\psi_{j,\varepsilon}(x_j) = \varepsilon^{-n}\psi_j(-x_j/\varepsilon)$ we have

$$\begin{aligned}&\iint K_\varepsilon(x_1, x_2)\phi_1(x_1)\phi_2(x_2)\, dx_1\, dx_2 \\ &\quad = \iint \langle \mathscr{K}\check\psi_{2,\varepsilon}(\cdot - x_2)\phi_2(x_2), \check\psi_{1,\varepsilon}(\cdot - x_1)\phi_1(x_1)\rangle\, dx_1\, dx_2.\end{aligned}$$

Replacing the integral by a Riemann sum first we conclude as in the proof of Lemma 4.1.3 that the integration can performed "under the sign", hence

$$\langle K_\varepsilon, \phi_1 \otimes \phi_2\rangle = \langle \mathscr{K}(\phi_2 * \check\psi_{2,\varepsilon}), \phi_1 * \check\psi_{1,\varepsilon}\rangle.$$

Since $\phi_j * \check\psi_{j,\varepsilon} \to \phi_j$ in $C^\infty_0(Y_j)$ when $\varepsilon \to 0$, it follows from (5.2.2) that the right-hand side converges to $\langle \mathscr{K}\phi_2, \phi_1\rangle$ when $\varepsilon \to 0$. Thus

$$\langle K_0, \phi_1 \otimes \phi_2\rangle = \langle \mathscr{K}\phi_2, \phi_1\rangle \quad \text{if } \phi_j \in C^\infty_0(Y_j),$$

and since Y_j are arbitrary relatively compact subsets of X_j, this completes the proof.

Example 5.2.2. The kernel of the identity map $\mathscr{K}: C_0^\infty(X) \to C_0^\infty(X)$, where X is an open set in $\mathbb{R}^n$, is the distribution

$$\langle K, \Phi \rangle = \int \Phi(x, x)\, dx, \qquad \Phi \in C_0^\infty(X \times X),$$

with support in the diagonal $\{(x, x), x \in X\}$.

Theorem 5.2.3. *The kernel of a continuous map* $\mathscr{K}: C_0^\infty(X) \to \mathscr{D}'(X)$ *is supported by the diagonal if and only if*

$$\mathscr{K}\phi = \sum a_\alpha \partial^\alpha \phi \tag{5.2.6}$$

where $a_\alpha \in \mathscr{D}'(X)$ *and the sum is locally finite.*

Proof. For the operator (5.2.6) we have

$$\langle \mathscr{K}\phi, \psi \rangle = \sum \langle a_\alpha, (\partial^\alpha \phi)\psi \rangle$$

so the kernel is given by

$$\langle K, \Phi \rangle = \sum \langle a_\alpha, \partial_y^\alpha \Phi(x, y)|_{x=y} \rangle$$

which is obviously supported by the diagonal. Conversely, if the kernel K of $\mathscr{K}$ is supported by the diagonal, it follows from Theorem 2.3.5 that K has the preceding form, which proves the theorem.

The preceding operators preserve supports; more generally, we have

Theorem 5.2.4. *If* $K \in \mathscr{D}'(X_1 \times X_2)$ *and* $\mathscr{K}$ *is the corresponding operator, then*

$$\operatorname{supp} \mathscr{K}u \subset \operatorname{supp} K \circ \operatorname{supp} u, \qquad u \in C_0^\infty(X_2). \tag{5.2.7}$$

Here $\operatorname{supp} K \subset X_1 \times X_2$ is considered as a relation acting on $\operatorname{supp} u \subset X_2$. Thus

$$\operatorname{supp} K \circ M = \{x_1 \in X_1;\ \exists x_2 \in M, (x_1, x_2) \in \operatorname{supp} K\}.$$

This is a closed set when M is compact, for $\operatorname{supp} K$ is closed.

Proof. Assume that $x_1 \notin \operatorname{supp} K \circ \operatorname{supp} u$. Then there is a neighborhood V of x_1 such that $V \cap (\operatorname{supp} K \circ \operatorname{supp} u) = \emptyset$. If $v \in C_0^\infty(V)$ then

$$(\operatorname{supp} v \otimes u) \cap \operatorname{supp} K = \emptyset$$

which proves that $\langle \mathscr{K}u, v \rangle = 0$, hence $\mathscr{K}u = 0$ in V.

Example 5.2.5. If $f: X_1 \to X_2$ is a continuous map and $\mathscr{K}\phi = \phi \circ f$, $\phi \in C_0^\infty(X_2)$, then the kernel is given by

$$\langle K, \Phi \rangle = \int \Phi(x, f(x))\, dx, \qquad \Phi \in C_0^\infty(X_1 \times X_2),$$

so the support is in the graph of f.

The operator (5.2.6) has a natural extension to all $\phi \in \mathscr{E}'$ if the coefficients $a_\alpha \in C^\infty$. General sufficient smoothness conditions for the

existence of such extensions will be given in Chapter VIII, but we give an elementary example now:

Theorem 5.2.6. *If* $K \in C^\infty(X_1 \times X_2)$ *then the map* $\mathscr{K}$ *defined by* (5.2.1) *has a continuous extension from* $\mathscr{E}'(X_2)$ *to* $C^\infty(X_1)$,

(5.2.8) $$\mathscr{K} u(x_1) = u(K(x_1, .)), \qquad u \in \mathscr{E}'(X_2),\ x_1 \in X_1.$$

Conversely, every continuous linear map $\mathscr{K}$ *from* $\mathscr{E}'(X_2)$ *to* $C^\infty(X_1)$ *is defined in this way by a kernel* $K \in C^\infty(X_1 \times X_2)$.

Proof. If $K \in C^\infty$ it follows from Theorem 2.1.3 that (5.2.8) defines a map $\mathscr{E}'(X_2) \to C^\infty(X_1)$, and the continuity is a consequence of Theorem 2.1.8. Conversely, if we are given a continuous map $\mathscr{K}$: $\mathscr{E}'(X_2) \to C^\infty(X_1)$ then

$$K(., x_2) = \mathscr{K} \delta_{x_2}, \qquad x_2 \in X_2,$$

is a continuous function of x_2 with values in $C^\infty(X_1)$. Taking difference quotients we find that K is continuously differentiable in x_2,

$$\langle y, \partial_{x_2} \rangle K(., x_2) = -\mathscr{K} \langle y, \partial \rangle \delta_{x_2}.$$

Repeating the argument gives $K \in C^\infty(X_1 \times X_2)$. We obtain (5.2.8) since this is true for finite linear combinations of Dirac measures and they are dense in $\mathscr{E}'$.

Notes

The tensor product was defined in Schwartz [1], and the kernel theorem was announced shortly afterwards in Schwartz [2]. In both cases the main point is the decomposition of test functions in $X_1 \times X_2$ into sums of tensor products of test functions in X_1 and in X_2. Thus the topological tensor product of $C_0^\infty(X_1)$ and $C_0^\infty(X_2)$ is involved, and Schwartz [4] gave a proof emphasizing this aspect. Ehrenpreis [4] published a more elementary proof where the decomposition was made by Fourier series expansion (see also Gask [1]). Here we have used instead the fact that a regularization of any test function in $X_1 \times X_2$ by a product of two test functions in the corresponding spaces $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ can be considered as a superposition of tensor products of test functions in X_1 and in X_2. (To be able to use this argument we had to define convolution before the tensor product.)

There is an interesting addendum to Theorem 5.2.3 due to Peetre [2]: If $\mathscr{K}$ is any linear map $C_0^\infty(X) \to C^\infty(X)$ with $\operatorname{supp} \mathscr{K} u \subset \operatorname{supp} u$, $u \in C_0^\infty(X)$, then $\mathscr{K}$ is a differential operator with C^∞ coefficients, that is, (5.2.6) is valid with $a_\alpha \in C^\infty$. Note that no continuity is assumed; it follows from the restriction on the supports.

Theorem 6.1.2. *Let $X_j \subset \mathbb{R}^{n_j}$, $j=1,2$, be open sets, and $f\colon X_1 \to X_2$ a C^∞ map such that $f'(x)$ is surjective for every $x \in X_1$. Then there is a unique continuous linear map $f^*\colon \mathscr{D}'(X_2) \to \mathscr{D}'(X_1)$ such that $f^*u = u \circ f$ when $u \in C^0(X_2)$. It maps $\mathscr{D}'^k(X_2)$ into $\mathscr{D}'^k(X_1)$ for every k. One calls f^*u the pullback of u by f.*

Proof. As already observed, the uniqueness follows from Theorem 4.1.5. To prove the existence we choose for any fixed $x_0 \in X_1$ a C^∞ map $g\colon X_1 \to \mathbb{R}^{n_1-n_2}$, for example a linear map, such that the direct sum $f \oplus g$,

$$X_1 \ni x \to (f(x), g(x)) \in \mathbb{R}^{n_1} = \mathbb{R}^{n_2} \oplus \mathbb{R}^{n_1-n_2}$$

has a bijective differential at x_0. By the inverse function theorem there is an open neighborhood $Y_1 \subset X_1$ of x_0 such that the restriction of $f \oplus g$ to Y_1 is a diffeomorphism on an open neighborhood Y_2 of $(f(x_0), g(x_0))$. We denote the inverse by h. If $u \in C^0(X_2)$ and $\phi \in C_0^\infty(Y_1)$ then a change of variables gives

$$\int (f^*u)\phi\, dx = \int u(f(x))\phi(x)\, dx = \int u(y')\phi(h(y))\,|\det h'(y)|\, dy$$

where we have written $y=(y', y'') \in \mathbb{R}^{n_2} \oplus \mathbb{R}^{n_1-n_2}$. Hence

$$(6.1.1) \qquad (f^*u)(\phi) = (u \otimes 1)(\Phi), \qquad \Phi(y) = \phi(h(y))\,|\det h'(y)|.$$

Here 1 is the function 1 in $\mathbb{R}^{n_1-n_2}$. If $u \in \mathscr{D}'(X_2)$ and we choose $u_j \in C_0^\infty(X_2)$ so that $u_j \to u$ in $\mathscr{D}'(X_2)$, it follows in view of the remark after Theorem 2.2.4 that f^*u_j converges in $\mathscr{D}'(X)$ to a distribution f^*u defined by (6.1.1) in Y_1. Thus (6.1.1) gives a local definition of f^*u, and the continuity of the map $u \to f^*u$ follows at once from (6.1.1). The theorem is proved.

Remark. The proof shows that if $f \in C^{k+1}$ only, then f^* is well defined and continuous in $\mathscr{D}'^k$. In fact, $\phi \to \Phi$ is continuous from C_0^k to C_0^k. (We need an extra derivative for f since $\det h'$ involves one derivative of f.)

Since we have defined f^*u by continuous extension from the case of functions u, it is clear that the usual rules of computation remain valid:

$$(6.1.2) \qquad \partial_j f^*u = \sum \partial_j f_k f^* \partial_k u, \qquad u \in \mathscr{D}'(X_2) \quad \text{(the chain rule)},$$

$$(6.1.3) \qquad f^*(au) = (f^*a)(f^*u); \qquad a \in C^\infty(X_2),\ u \in \mathscr{D}'(X_2).$$

Here f is assumed to satisfy the hypotheses of Theorem 6.1.2. If in addition we have a C^∞ map $g\colon X_2 \to X_3$ with surjective differential, then

$$(6.1.4) \qquad (g \circ f)^*u = f^*g^*u, \qquad u \in \mathscr{D}'(X_3),$$

In practice it is often convenient to use the notation $u(f)$, $u \circ f$ or even $u(f(x))$ instead of f^*u since (6.1.2)–(6.1.4) look more familiar

then. However, one must always keep in mind then that one is referring to an extension of the pointwise definition given by (6.1.1).

Example 6.1.3. If f is a diffeomorphism $X_1 \to X_2$ between open sets in $\mathbb{R}^n$ then $f^*\delta_y = |\det f'(x)|^{-1}\delta_x$ where $f(x)=y$. This follows from (6.1.1) with $h=f^{-1}$.

Example 6.1.4. If $M_t x = tx$, $x \in \mathbb{R}^n$, $t>0$, then

$$(M_t^* u)(\phi) = u(\phi(\,.\,/t)/t^n).$$

Thus (3.2.18) with t replaced by $1/t$ means that

$$M_t^* u = t^a u \qquad \text{in } \mathbb{R}^n \setminus 0,$$

which is just the usual definition of homogeneity of degree a.

Theorem 6.1.5. *If ρ is a real valued function in $C^\infty(X)$, $X \subset \mathbb{R}^n$, and if $|\rho'| = (\sum |\partial\rho/\partial x_j|^2)^{\frac{1}{2}} \neq 0$ when $\rho=0$, then $\rho^*\delta_0 = dS/|\rho'|$ where dS is the Euclidean surface measure on the surface $\{x;\ \rho(x)=0\}$.*

Proof. Let $\rho(x_0)=0$ and assume for example that $\partial\rho(x_0)/\partial x_1 \neq 0$. Then we can apply (6.1.1) in a neighborhood with

$$h^{-1}(x) = (\rho(x), x_2, \dots, x_n).$$

Then $h(0, y_2, \dots, y_n) = (\psi(y_2, \dots, y_n), y_2, \dots, y_n)$ lies on the surface $\rho=0$, and we have for $\phi \in C_0^\infty(Y)$ if Y is a small neighborhood of x_0

$$\langle \rho^*\delta_0, \phi\rangle = \int (\phi/|\partial_1\rho|) \circ h(0, y_2, \dots, y_n)\, dy_2 \dots dy_n.$$

Since $\rho(\psi, y_2, \dots, y_n)=0$ we have for $j=2, \dots, n$

$$\partial_1\rho\,\partial\psi/\partial y_j + \partial_j\rho = 0.$$

Hence

$$|\rho'| = |\partial_1\rho|\,M, \qquad M = \Big(1 + \sum_2^n (\partial\psi/\partial y_j)^2\Big)^{\frac{1}{2}}.$$

Since $dS = M\,dy_2 \dots dy_n$ with the parameters $y_2 \dots y_n$, this proves the theorem.

From (6.1.2) it follows that if H is the Heaviside function then

$$\partial_j \rho^* H = (\partial_j\rho)\rho^*\delta_0 = (\partial_j\rho)/|\rho'|\,dS,$$

which means that we have given another proof of the Gauss-Green formula (3.1.5). One calls $\rho^*\delta_0$ a *simple layer* on the surface $\rho=0$, and its derivatives are called *multiple layers.* They are essentially the pull-backs by ρ of the derivatives of δ_0. In fact, let

$\mathscr{D}'(X)$ is obviously a vector space with the natural definition of addition and multiplication by complex numbers,

$$(a_1 u_1 + a_2 u_2)(\phi) = a_1 u_1(\phi) + a_2 u_2(\phi);$$
$$\phi \in C_0^\infty(X), \quad u_j \in \mathscr{D}'(X), \quad a_j \in \mathbb{C}.$$

We shall always use the *weak topology* in $\mathscr{D}'(X)$ (also called the weak* topology), that is, the topology defined by the semi-norms

$$\mathscr{D}'(X) \ni u \to |u(\phi)|,$$

where ϕ is any fixed element of $C_0^\infty(X)$. Thus $u_i \to u$ means that

$$u_i(\phi) \to u(\phi)$$

for every $\phi \in C_0^\infty(X)$. Occasionally we shall need the following completeness property:

Theorem 2.1.8. *If u_j is a sequence in $\mathscr{D}'(X)$ and*

$$u(\phi) = \lim_{j \to \infty} u_j(\phi) \tag{2.1.7}$$

exists for every $\phi \in C_0^\infty(X)$, then $u \in \mathscr{D}'(X)$. Thus $u_j \to u$ in $\mathscr{D}'(X)$ as $j \to \infty$. Moreover, (2.1.2) *is valid for all u_j with constants C and k independent of j, and $u_j(\phi_j) \to u(\phi)$ if $\phi_j \to \phi$ in $C_0^\infty(X)$.*

Proof. When K is a compact subset of X the space $C_0^\infty(K)$ is a Fréchet space with the topology defined by the semi-norms

$$\|\phi\|_\alpha = \sup |\partial^\alpha \phi|, \quad \phi \in C_0^\infty(K).$$

(The completeness is a consequence of Theorem 1.1.5.) (2.1.2) is valid for u_j (with constants C and k which may a priori depend on j), so u_j restricted to $C_0^\infty(K)$ is a continuous linear form on $C_0^\infty(K)$. For fixed $\phi \in C_0^\infty(K)$ it follows from (2.1.7) that the sequence $u_j(\phi)$ is bounded. Hence the principle of uniform boundedness (the Banach-Steinhaus theorem) shows that (2.1.2) is valid for all u_j with constants C and k independent of j. When $j \to \infty$ we obtain (2.1.2) for the limit u. If $\phi_j \to \phi$ in $C_0^\infty(X)$ we have $\operatorname{supp} \phi_j \subset K$ for some compact subset K of X and all j. Hence $u_j(\phi_j - \phi) \to 0$ by the uniformity of (2.1.2), which proves that $u_j(\phi_j) \to u(\phi)$.

By Cauchy's convergence principle for $\mathbb{C}$ the existence of the limit (2.1.7) means precisely that $u_j(\phi) - u_k(\phi) \to 0$ when $j,\ k \to \infty$. Hence an equivalent statement of Theorem 2.1.8 is that every sequence u_j in $\mathscr{D}'(X)$ such that $u_j - u_k \to 0$ when $j,\ k \to \infty$ must have a limit u in $\mathscr{D}'(X)$.