1 Weak derivatives

The purpose of these notes is to give a very basic introduction to Sobolev spaces. More extensive treatments can e.g. be found in the classical references [1] and [3]. Sobolev spaces are natural generalisations of the Lebesgue spaces $L^p$. Simply stated, if $\Omega \subseteq \mathbb{R}^d$ is an open set, $1 \leq p \leq \infty$, and $m$ is a positive integer, the Sobolev space $W^{m,p}(\Omega)$ consists of functions (or equivalence classes of functions) in $L^p(\Omega)$ whose partial derivatives up to order $m$ are in $L^p(\Omega)$. We will start by giving a more precise definition of what we mean by ‘partial derivatives’, since these are not defined in the classical sense. Throughout these notes, $dx$ refers to Lebesgue measure on $\mathbb{R}^d$ and we assume that all functions are real-valued (analogous results hold for complex-valued functions). We will also identify elements of $L^p$ with functions.

Before discussing weak derivatives, we introduce some notation. The class of continuous, real-valued functions on $\Omega$ with compact support is denoted $C_c(\Omega)$ and the subset of smooth functions is denoted $C_{c}^\infty(\Omega)$.¹ Some results concerning approximation of $L^p$-functions by functions in $C_{c}^\infty$ are collected in the appendix. We denote by $L^p_{\text{loc}}(\Omega)$ the set of measurable functions $f$ such that $f|_K \in L^p(K)$ for any compact subset $K$ of $\Omega$. Note that $L^p_{\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$ whenever $1 \leq p \leq \infty$.

If $f$ and $\varphi$ are $C^1$ functions on $\Omega$ and $\varphi$ has compact support, then

$$\int_{\Omega} f \partial_k \varphi \, dx = - \int_{\Omega} \partial_k f \varphi \, dx,$$

where $\partial_k = \partial/\partial x_k$. This can e.g. be proved using Fubini’s theorem and the usual integration by parts formula in one dimension. We take this as our definition of weak derivatives.

**Definition.** Let $f \in L^1_{\text{loc}}(\Omega)$. We say that $f$ is **weakly partially differentiable with respect to $x_k$** if there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_{\Omega} f \partial_k \varphi \, dx = - \int_{\Omega} g \varphi \, dx$$

¹The notation $C_0(\Omega)$ and $C_0^\infty(\Omega)$ is also common.
for every \( \varphi \in C_c^\infty(\Omega) \). If \( f \) is weakly partially differentiable with respect to \( x_k \), we call \( g \) the \textit{weak partial derivative of \( f \) with respect to \( x_k \)} and denote it by \( \partial_k f \).

The fact that we took \( \varphi \) to be \( C^\infty \) instead of just \( C^1 \) in the definition is not essential, but is convenient in some situations. It’s not difficult to show that if \( f \) is weakly partially differentiable, then the integration by parts formula also holds if \( \varphi \) is only \( C^1 \).

For the above definition to make sense, we should prove that \( g \) is unique (up to a.e. equality).

**Proposition 1** (Uniqueness of weak derivatives). Assume that \( g \) and \( h \) are both weak partial derivatives with respect to \( x_k \) of \( f \in L^1_{\text{loc}}(\Omega) \). Then \( g = h \) a.e.

**Proof.** By linearity we have that

\[
\int_\Omega (g - h) \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\]

Setting \( u = g - h \) in \( \Omega \) and \( u \equiv 0 \) in \( \Omega^c \), we find that

\[
\int_{\mathbb{R}^d} u \varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d) \text{ such that } \text{supp } \varphi \subseteq \Omega.
\]

Hence, if \( x \in \Omega \), we have that

\[
\int_{\mathbb{R}^d} u(y) J_\varepsilon(x - y) \, dy = 0
\]

for sufficiently small \( \varepsilon > 0 \), where \( J_\varepsilon \) is the family of mollifiers constructed in the appendix (\( J_\varepsilon(x - \cdot) \) is supported in \( B_\varepsilon(x) \subset \Omega \) if \( \varepsilon > 0 \) is small). Letting \( \varepsilon \to 0^+ \) we obtain by Theorem 12 in the appendix that \( u = 0 \) a.e in \( \Omega \), which concludes the proof.

**Example.** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = |x| \). Then it’s clear that \( f \in L^1_{\text{loc}}(\mathbb{R}) \). Moreover, if \( \varphi \in C_c^\infty(\mathbb{R}) \) with \( \text{supp } \varphi \subseteq [a, b] \), \( a < 0 < b \), then

\[
\int_{\mathbb{R}} |x| \varphi'(x) \, dx = \int_0^b x \varphi'(x) \, dx - \int_a^0 x \varphi'(x) \, dx
\]

\[
= -\int_0^b \varphi(x) \, dx + \int_a^0 \varphi(x) \, dx + [x \varphi(x)]_0^b - [x \varphi(x)]_a^0
\]

\[
= -\int_{\mathbb{R}} \text{sgn}(x) \varphi(x) \, dx.
\]

Hence, the weak derivative of \( |x| \) is

\[
\text{sgn}(x) = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0.
\end{cases}
\]

Note that the weak derivative in this example is simply the pointwise derivative wherever it exists. This will be generalised in Section 3.
2 Sobolev spaces

In this section we define Sobolev spaces and discuss some of their basic properties. For simplicity we restrict ourselves to Sobolev spaces of order one, although the definitions and results are easily generalised to higher order spaces.

**Definition.** Let $\Omega \subseteq \mathbb{R}^d$ an open, non-empty set and let $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined as the set of $f \in L^p(\Omega)$ such that all the weak derivatives $\partial^k f, k = 1, \ldots, d$, exist and belong to $L^p(\Omega)$. We equip it with the norm

$$\|f\|_{1,p} = \left( \|f\|_p^p + \sum_{k=1}^d \|\partial^k f\|_p^p \right)^{\frac{1}{p}}$$

when $1 \leq p < \infty$, and

$$\|f\|_{1,\infty} = \|f\|_\infty + \sum_{k=1}^d \|\partial^k f\|_\infty$$

when $p = \infty$.  

There are several equivalent ways of defining the norm (in the sense that they generate the same topology). When $p = 2$,

$$(f, g) = \int_\Omega \left( fg + \sum_{k=1}^d \partial^k f \partial^k g \right) \, dx$$

defines an inner product on $W^{1,2}(\Omega)$ and $\| \cdot \|_{1,p}$ is the induced norm with our choice of definition. This space is so important that it has a special notation: $H^1(\Omega) = W^{1,2}(\Omega)$.

**Theorem 2.** $W^{1,p}(\Omega)$ is complete for any $p \in [1, +\infty]$. In particular, $H^1(\Omega)$ is a Hilbert space.

**Proof.** Let $\{f_n\}$ be a Cauchy sequence in $W^{1,p}(\Omega)$. Then $\{f_n\}$ is also a Cauchy sequence in $L^p(\Omega)$. Hence, it converges to some $f$ in $L^p(\Omega)$. Similarly, all the partial derivatives $\{\partial^k f_n\}$ converge to some function $g_k$ in $L^p(\Omega)$. It suffices to prove that $g_k$ is the weak partial derivative of $f$ with respect to $x_k$. But this follows by taking limits in the identity

$$\int_\Omega f_n \partial^k \varphi \, dx = - \int_\Omega \partial^k f_n \varphi \, dx, \quad \varphi \in C^\infty_c(\Omega),$$

which is possible by Hölder’s inequality since $\varphi$ and $\partial^k \varphi$ belong to $L^q(\Omega)$, where $1/p + 1/q = 1$.  

We end this section with a natural approximation result.

**Theorem 3.** $C^\infty_c(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$ whenever $1 \leq p < \infty$. 


Proof. Let \( f \in W^{1,p}(\mathbb{R}^d) \) and let \( J_\varepsilon \) be as in the appendix. Using the fact that \( \partial_k(J_\varepsilon * f) = J_\varepsilon * \partial_k f \) (differentiate under the integral sign and move the derivative to \( f \) using the definition of weak derivatives), we obtain that \( f_\varepsilon = J_\varepsilon * f \) is in \( W^{1,p}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) with \( f_\varepsilon \to f \) as \( \varepsilon \to 0^+ \). The only problem is that \( J_\varepsilon f \) doesn’t have compact support. Now let \( \varphi \in C^\infty(\mathbb{R}^d) \) be a cut-off function as in Corollary 14. Then \( \varphi(x/n) \) is supported in the ball of radius \( 2n \) and is equal to one in the ball of radius \( n \). We have that

\[
\|\varphi(\cdot/n)f_\varepsilon - f_\varepsilon\|_p^p \leq \int_{|x|\geq n} |f_\varepsilon(x)|^p \, dx \to 0
\]
as \( n \to \infty \). Moreover, \( \partial_{x_k}\varphi(x/n)f_\varepsilon(x) = \frac{1}{n}(\partial_k\varphi)(x/n)f_\varepsilon(x) + \varphi(x/n)\partial_k f_\varepsilon(x) \). The second term converges to \( f_\varepsilon \) in \( L^p \) as \( n \to \infty \) by the previous argument, whereas the first converges to 0 since \( \|\partial_k\varphi(x/n)f_\varepsilon(x)\|_p \leq \|\partial_k\varphi\|_{\infty} \|f_\varepsilon(x)\|_p \). The result follows by first taking \( \varepsilon \) sufficiently small and then taking \( n \) sufficiently large. \( \Box \)

3 Relation to absolutely continuous functions

If \( I \subseteq \mathbb{R} \) is an interval, we let \( \text{AC}(I) \) be the class of real-valued absolutely continuous functions \( f: I \to \mathbb{R} \). Recall that \( f \in \text{AC}([a,b]) \) if and only if \( f'(x) \) exists for a.e. \( x \in [a,b] \), \( f' \) is integrable on \([a,b] \) and \( f \) can be recovered from \( f' \) by the formula

\[
(1) \quad f(x) = f(a) + \int_a^x f'(x) \, dx
\]

(we can of course replace \( a \) in this formula by any other point in \([a,b]\)). If \( f \) is absolutely continuous it is also continuous. Similarly, we let \( \text{AC}_{\text{loc}}(I) \) be the class of locally absolutely continuous functions, where \( f: I \to \mathbb{R} \) is locally absolutely continuous on \( I \) if it’s absolutely continuous on any compact subinterval \([a,b] \subset I \). Note that \( \text{AC}_{\text{loc}}([a,b]) = \text{AC}([a,b]) \) if \([a,b]\) is a compact interval. It’s straightforward to see that \( f \in \text{AC}_{\text{loc}}(I) \) if and only if \( f \) is differentiable a.e. on \( I \), \( f' \in L^1_{\text{loc}}(I) \) and \( f \) can be recovered from \( f' \) from the formula (1) where \( a \) is an arbitrary point in \( I \).

The goal of this section is to relate absolutely continuous functions to Sobolev functions, but we first need a lemma.

Lemma 4. Assume that \( f \in L^1_{\text{loc}}(\mathbb{R}) \) has weak derivative 0. Then \( f \) is constant (a.e.).

Proof. Suppose that \( \varphi \in C^\infty_c(\mathbb{R}) \). Then \( \int_{-\infty}^x \varphi(y) \, dy \) is smooth, vanishes when \( x \leq a \), for some \( a \in \mathbb{R} \), and is constantly equal to \( \int_{\mathbb{R}} \varphi \, dx \) when \( x \geq b \), for some \( b \in \mathbb{R} \). Hence, it belongs to \( C^\infty_c(\mathbb{R}) \) if and only if \( \int_{\mathbb{R}} \varphi \, dx = 0 \)

Let \( \varphi_0 \) be a fixed function in \( C^\infty_c(\mathbb{R}) \) with \( \int_{\mathbb{R}} \varphi_0 \, dx = 1 \). For a general \( \varphi \in C^\infty_c(\mathbb{R}) \), we let \( I(\varphi) = \int_{\mathbb{R}} \varphi \, dx \). Then

\[
I(\varphi - I(\varphi)\varphi_0) = I(\varphi) - I(\varphi)I(\varphi_0) = 0.
\]
Hence, $\psi(x) = \int_{-\infty}^{x} (\varphi(y) - I(\varphi)\varphi_0(y)) \, dy$ belongs to $C_\infty^\infty(\mathbb{R})$.

By assumption we have that

$$0 = \int_{\mathbb{R}} f' \psi \, dx = \int_{\mathbb{R}} f(\varphi - I(\varphi)\varphi_0) \, dx$$

$$= \int_{\mathbb{R}} f \varphi \, dx - \left( \int_{\mathbb{R}} \varphi \, dx \right) \left( \int_{\mathbb{R}} f \varphi_0 \, dx \right)$$

$$= \int_{\mathbb{R}} (f - I(f\varphi_0))\varphi \, dx.$$

Since this holds for all $\varphi \in C_\infty^\infty(\mathbb{R})$, we find as in the proof of Proposition 1 that $f = I(f\varphi_0)$ a.e. and this concludes the proof.

**Theorem 5.** $f \in W^{1,p}(\mathbb{R})$ if and only if $f \in AC_{loc}(\mathbb{R})$ with $f', f'' \in L^p(\mathbb{R})$, where $f'$ is the a.e. pointwise derivative of $f$.

Remark: in the above statement we mean that there is a representative of $f \in W^{1,p}$ which is locally absolutely continuous.

**Proof.** $f \in AC_{loc}(\mathbb{R})$ with $f, f' \in L^p(\mathbb{R})$, then using the integration by parts formula for absolutely continuous functions (see [2, Cor. 6.3.9]), we obtain that $f$ has weak derivative $f' \in L^p(\mathbb{R})$.

Assume instead that $f \in W^{1,p}(\mathbb{R})$. Let $g$ be the weak derivative of $f$ and set

$$G(x) = \int_{0}^{x} g(y) \, dy.$$ 

Then $G$ is locally absolutely continuous with $G'(x) = g(x)$ a.e. Hence $G$ and $f$ both have weak derivative $g$. From Lemma 4 we obtain that $G - f$ is constant a.e. It follows that $f \in AC_{loc}(\mathbb{R})$ with $f' = g$.

One can show a similar result in the case $d \geq 2$ where absolute continuity is replaced by absolute continuity on a.e. line parallel to the coordinate axes (see [3]).

### 4 Sobolev embedding theorems

In Section 3 we saw that if $f \in W^{1,p}(\mathbb{R})$, for some $p \in [1, +\infty]$, then $f$ is in fact absolutely continuous. In particular, it is continuous. One can actually show quite a bit more.

Recall that a function $f: \mathbb{R}^d \to \mathbb{R}$ is called (uniformly) **Lipschitz continuous** if there exists a constant $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y| \text{ for all } x, y \in \mathbb{R}^d.$$
Similarly, \( f \) is called (uniformly) Hölder continuous with exponent \( \alpha \in (0, 1) \) if there is a constant \( C \geq 0 \) such that
\[
|f(x) - f(y)| \leq C|x - y|^\alpha \text{ for all } x, y \in \mathbb{R}^d.
\]

For simplicity, we restrict ourselves to one dimension in these notes. Throughout this section we identify an element \( f \in W^{1,p}(\mathbb{R}) \) with its locally absolutely continuous representative. We begin by discussing the case \( p = \infty \).

**Proposition 6.** If \( f \in W^{1,\infty}(\mathbb{R}) \), then \( f \) is Lipschitz continuous.

*Proof.* This follows from the calculation
\[
|f(x) - f(y)| = \left| \int_y^x f'(t) \, dt \right| \leq \|f'\|_{\infty}|x - y|.
\]

One can actually show a converse: if \( f \) is bounded and Lipschitz, then \( f \in W^{1,\infty}(\mathbb{R}) \) (see [3]).

We next consider the case \( 1 < p < \infty \).

**Proposition 7.** Let \( 1 < p < \infty \). If \( f \in W^{1,p}(\mathbb{R}) \), then \( f \) is Hölder continuous with exponent \( 1 - 1/p \). Moreover, \( f \) is bounded with \( \lim_{x \to \pm \infty} f(x) = 0 \).

*Proof.* Let \( q \) be the conjugate exponent of \( p \) and \( \alpha = 1/q \). Then \( \alpha = 1 - 1/p \) and
\[
|f(x) - f(y)| = \left| \int_y^x f'(t) \, dt \right| \leq \|f'\|_p|x - y|^{\alpha}
\]
by Hölder’s inequality. This shows that \( f \) is Hölder continuous.

Let \( f \in C_c^\infty(\mathbb{R}) \). Note that \( |f|^p \) is continuously differentiable and that
\[
\frac{d}{dx}|f(x)|^p = p|f(x)|^{p-1} \text{sgn}(f(x))f'(x).
\]

Integrating this equality from \(-\infty\) to \( x \) and using Young’s inequality, we obtain that
\[
|f(x)|^p \leq p \int_{-\infty}^x |f(y)|^{p-1}|f'(y)| \, dy
\]
\[
\leq p \int_{\mathbb{R}} \left( \frac{|f(y)|^{q(p-1)}}{q} + \frac{|f'(y)|^p}{p} \right) \, dy
\]
\[
\leq B\|f\|_{1,p}^p,
\]
where \( B = \max\{1, p/q\} \). Hence,
\[
(2) \quad \|f\|_{\infty} \leq C\|f\|_{1,p}
\]
with \( C = B^{1/p} \).
Now let $f \in W^{1,p}(\mathbb{R})$ and let $\{f_n\} \in C^\infty_c(\mathbb{R})$ with $f_n \to f$ in $W^{1,p}$. Then
\[ \|f_n - f_m\|_\infty \leq C \|f_n - f_m\|_{1,p}, \]
so $f_n$ converges uniformly to some continuous and bounded function $g$. Since $f_n \to f$ in $L^p$ we must have $g = f$ (a.e.) and since $\|f_n\|_\infty \to \|f\|_\infty$ while $\|f_n\|_{1,p} \to \|f\|_{1,p}$, we obtain that (2) also holds for $f$. Let $\epsilon > 0$ and choose $n$ such that $\|f - f_n\|_\infty < \epsilon$. Choose $R$ such that $f_n(x) = 0$ for $|x| \geq R$. Then
\[ |f(x)| = |f(x) - f_n(x)| < \epsilon \]
when $|x| \geq R$. Hence, $\lim_{x \to \pm \infty} f(x) = 0$. \hfill \(\square\)

When $p = 1$, one can still show that following result, whose proof we leave to the reader.

**Proposition 8.** If $f \in W^{1,1}(\mathbb{R})$, then $f$ is absolutely continuous, of finite variation and vanishes at $\pm \infty$.

The results above are examples of Sobolev embedding theorems, showing that certain Sobolev spaces are embedded in other spaces. Similar results hold for $d \geq 2$ and for Sobolev spaces of higher order (see [1, 3]). In particular, these are useful in the study of partial differential equations, where they can be used to show that weak solutions are in fact classical solutions (see the next section).

## 5 An application

Weak derivatives and Sobolev spaces are very useful for studying partial differential equations. The idea is that it allows one to separate the questions of existence and regularity. One starts by reformulating the equation in a ‘weak’ sense and proves the existence of a ‘weak solution’ (which a priori doesn’t have enough regularity to satisfy the equation in the classical sense). One then shows that the solution in fact has enough regularity to satisfy the equation in the classical sense.

As an example we consider the (elliptic) partial differential equation
\[ -\Delta f + f = g, \]
where $\Delta = \partial^2_1 + \cdots + \partial^2_d$ is the Laplace operator. Here $g$ is a given function and we want to find a solution $f$. For simplicity we consider the equation in $\mathbb{R}^d$, although one can also consider it in some domain $\Omega \subset \mathbb{R}^d$ with suitable boundary conditions. In order to find a solution, we begin by relaxing what we mean by a solution. Assume that $g \in L^2(\mathbb{R}^d)$. Multiplying the equation by a test function $\varphi \in C^\infty_c(\mathbb{R}^d)$ and integrating by parts, we obtain that
\[ \int_{\mathbb{R}^d} (\nabla f \cdot \nabla \varphi + f \varphi) \, dx = \int_{\mathbb{R}^d} g \varphi \, dx. \]
Note that this equation makes sense even if $f$ is just in $H^1(\mathbb{R}^d)$. 

7
**Definition.** We say that \( f \in H^1(\mathbb{R}^d) \) is a weak solution of (3) if (4) holds for each \( \varphi \in C_\infty^\infty(\mathbb{R}^d) \).

We next note that \( f \in H^1(\mathbb{R}^d) \) is a weak solution if and only if (4) holds for each \( \varphi \in H^1(\mathbb{R}^d) \). Indeed, one direction is clear since \( C_\infty^\infty(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \) and the other direction follows by approximating \( \varphi \in H^1(\mathbb{R}^d) \) by a sequence of functions in \( C_\infty^\infty(\mathbb{R}^d) \). With this at hand, we can prove the following theorem.

**Theorem 9.** For each \( g \in L^2(\mathbb{R}^d) \), there exists a unique weak solution \( f \in H^1(\mathbb{R}^d) \) of (3).

**Proof.** Note that \( \varphi \mapsto \int_{\mathbb{R}^d} g\varphi \, dx \) is a bounded linear functional on \( H^1(\mathbb{R}^d) \) by Hölder’s inequality. Hence the Fréchet-Riesz representation theorem (see Exercise 3.5.7 in [2]) shows that there is a unique function \( f \in H^1(\mathbb{R}^d) \) such that

\[
(f, \varphi) = \int_{\mathbb{R}^d} g\varphi \, dx,
\]

where

\[
(f, \varphi) = \int_{\mathbb{R}^d} (f\varphi + \nabla f \cdot \nabla \varphi) \, dx
\]

is the \( H^1 \) inner product of \( f \) and \( \varphi \). But this means that \( f \) is a weak solution of (3).

One can in fact show that the above solution belongs to the space \( H^2(\mathbb{R}^d) \) consisting of functions \( f \in L^2(\mathbb{R}^d) \) having weak partial derivatives of all orders less than or equal to 2 which belong to \( L^2(\mathbb{R}^d) \) and that (3) holds a.e. This can e.g. be done using Fourier analysis. In general, one can show that if \( g \in H^n(\mathbb{R}^d) \), for some integer \( n \geq 1 \), then the solution \( f \in H^{n+2}(\mathbb{R}^d) \). Combining this with a suitable Sobolev embedding, one can show that \( f \) is \( C^2 \) and satisfies (3) in the classical sense if \( n \) is sufficiently large (depending on \( d \)). This is, however, somewhat unsatisfactory. One would guess that if \( g \) is continuous then \( f \) is a classical solution (i.e. \( f \in C^2 \) and satisfies the equation pointwise). Somewhat surprisingly perhaps, this turns out to be false in general. However, if one imposes a little bit more regularity and requires that \( g \) is Hölder continuous, then \( f \) is indeed a classical solution.

The previous result can also be obtained using Fourier analysis, but the above method has the advantage that it generalises easily to elliptic PDEs with variable coefficients and to other domains.

**Appendix**

This appendix contains some results on approximation by smooth functions. It is often convenient to be able to approximate an \( L^p \) function by a smooth function and in particular by a smooth function with compact support. Recall that a
function $f: \mathbb{R}^d \to \mathbb{R}$ is said to have *compact support* if it vanishes outside a compact set. The support of $f$ is defined as

$$\text{supp } f = \{ x : f(x) \neq 0 \}.$$ 

This definition isn’t very useful for $L^p$ functions, since the support can change if $f$ is altered on a set of measure zero. The *essential support*, $\text{ess supp } f$, as the smallest closed set outside of which $f = 0$ a.e., i.e.

$$\text{ess supp } f = \mathbb{R}^d \setminus \bigcup \{ U \subseteq \mathbb{R}^d : U \text{ is open and } f(x) = 0 \text{ for a.e. } x \in U \}.$$ 

Clearly $\text{ess supp } f = \text{ess supp } g$ if $f = g$ a.e. Moreover, if $f$ is continuous, then $\text{ess supp } f = \text{supp } f$ since then $f(x) = 0$ for a.e. $x \in U$ if and only if $f(x) = 0$ for all $x \in U$. In the future we will simply talk about the support of $f$, although what we really mean is the essential support.

The class of continuous, real-valued functions on $\mathbb{R}^d$ with compact support is denoted $C_c(\mathbb{R}^d)$. and the subset of smooth functions is denoted $C_c^\infty(\mathbb{R}^d)$ Before approximating with smooth functions, we recall that any $L^p$ function can be approximated by continuous functions with compact support.

**Lemma 10.** $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ whenever $1 \leq p < \infty$.

This is discussed for $d = 1$ in Ch. 3.4 of Cohn. The general case is proved in a similar way.

We should also show that $C_c^\infty(\mathbb{R}^d)$ is not empty, a fact which is not obvious.

**Lemma 11** (Existence of functions in $C_c^\infty(\mathbb{R}^d)$). There exists a non-negative function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp } \varphi = \overline{B_1(0)}$.

**Proof.** Note first that the function

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is $C^\infty$ on $\mathbb{R}$ with support $[0, \infty)$. Indeed, for $t > 0$ all the derivatives are polynomials in $1/t$ times $e^{-1/t}$ and $t^a e^{-1/t} \to 0$ as $t \to 0^+$ for any $a \in \mathbb{R}$. The function $\varphi$ defined by

$$\varphi(x) = f(1 - |x|^2)$$

satisfies the conditions.

The basic idea in the approximation argument is to use convolutions with smooth functions. Recall that the *convolution* $f * g$ of two measurable functions $f$ and $g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y) \, dy$$
whenever the integral is defined. We can think of this as a sort of average of $f$ with weight $g$. Thus, convolutions tend to ‘smear’ out at a function, making it smoother. Using the change of variables $y \mapsto x - y$, we find that

$$(f * g)(x) = (g * f)(x) = \int_{\mathbb{R}^d} f(y)g(x - y) \, dy.$$  

By Hölder’s inequality, we obtain that $f * g \in L^\infty$ if $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in [1, \infty]$. Moreover,

$$(5) \quad \|f * g\|_\infty \leq \|f\|_p \|g\|_q.$$  

From Minkowski’s inequality for integrals, we obtain that

$$(6) \quad \|f * g\|_p \leq \|f\|_p \|g\|_{1,1} \leq \|f\|_p \|g\|_{1, \infty},$$  

so that $f * g \in L^p$ if $f \in L^p$ and $g \in L^1$. If $f$ and $g$ have compact support, it also follows from the definition that

$$\text{supp}(f * g) \subseteq \text{supp} f + \text{supp} g.$$  

Finally, a very important property of convolutions is the following. If $g \in L^1(\mathbb{R}^d)$ and $f \in C^1(\mathbb{R}^d)$ is bounded with bounded partial derivatives, then $f * g \in C^1$ with

$$\partial_k (f * g) = \partial_k f * g.$$  

This follows from the definition of the convolution and differentiation under the integral sign.

We can now construct a smooth ‘approximation of the identity’ as follows. After dividing $\varphi$ in Lemma 11 by $\int_{\mathbb{R}^d} \varphi \, dx$, we can assume that $\int_{\mathbb{R}^d} \varphi \, dx = 1$. We set

$$J_\varepsilon(x) = \varepsilon^{-d} \varphi(\varepsilon^{-1} x), \quad \varepsilon > 0.$$  

Then $J_\varepsilon \in C^\infty_c(\mathbb{R}^d)$ with supp $J_\varepsilon = B_\varepsilon(0)$ and $\int_{\mathbb{R}^d} J_\varepsilon(x) \, dx = 1$ for all $\varepsilon > 0$ (change of variables). One can think of $J_\varepsilon$ as converging to a ‘unit mass’ at the origin. This can be made sense of using distribution theory. The family $\{J_\varepsilon\}_{\varepsilon > 0}$ is an example of mollifiers.

**Theorem 12.**

(i) If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $J_\varepsilon * f \in C^\infty(\mathbb{R}^d)$.

(ii) If $f \in L^1(\mathbb{R}^d)$ has compact support, then $J_\varepsilon * f \in C^\infty_c(\mathbb{R}^d)$.

(iii) If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $(J_\varepsilon * f)(x) \to f(x)$ whenever $x$ is a Lebesgue point of $f$. In particular, $J_\varepsilon * f \to f$ a.e.

(iv) If $f \in C(\mathbb{R}^d)$, then $J_\varepsilon * f \to f$ uniformly on any compact subset of $\mathbb{R}^d$.  

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(v) If \( f \in L^p(\mathbb{R}^d) \), where \( 1 \leq p < \infty \), then \( \|J_\varepsilon * f\|_p \leq \|f\|_p \) and \( \lim_{\varepsilon \to 0^+} \|J_\varepsilon * f - f\|_p \).

**Proof.** The first statement follows by repeated differentiation under the integral sign.

The second statement follows since \( \text{supp } J_\varepsilon \subseteq \text{supp } f + \text{supp } J_\varepsilon \) and both \( f \) and \( J_\varepsilon \) have compact supports.

We have that
\[
(J_\varepsilon * f)(x) - f(x) = \int_{\mathbb{R}^d} J_\varepsilon(x - y)(f(y) - f(x)) \, dy
\]
\[
= \varepsilon^{-d} \int_{B_\varepsilon(x)} J(\varepsilon^{-1}(x - y))(f(y) - f(x)) \, dy
\]
and therefore
\[
|(J_\varepsilon * f)(x) - f(x)| \leq \lambda(B_1(0))\|J\|_\infty \frac{1}{\lambda(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |f(y) - f(x)| \, dy \to 0
\]
if \( x \) is a Lebesgue point of \( f \). This proves (iii).

We have that
\[
(J_\varepsilon * f)(x) - f(x) = \int_{\mathbb{R}^d} J_\varepsilon(y)(f(x - y) - f(x)) \, dy
\]
\[
= \int_{B_1(0)} J(y)(f(x - \varepsilon y) - f(x)) \, dy
\]
Since \( f \) is uniformly continuous on any compact subset \( K \subset \mathbb{R}^d \), it follows that
\[
\sup_{(x,y) \in K \times B_1(0)} |f(x - \varepsilon y) - f(x)| \to 0 \text{ as } \varepsilon \to 0^+,
\]
proving (iv).

The first part of (v) is a consequence of (6). Let \( \delta > 0 \). Using Lemma 10 we can find \( g \in C_c(\mathbb{R}^d) \) such that \( \|f - g\|_p < \delta \) and it follows from (ii) that \( J_\varepsilon * g - g \) has support in a compact set which is independent of \( \varepsilon \) (sufficiently small) and hence from (iv) that \( J_\varepsilon * g \to g \) uniformly. It follows that \( \|J_\varepsilon * g - g\|_p < \delta \) if \( \varepsilon \) is sufficiently small. For \( \varepsilon \) sufficiently small, we therefore have
\[
\|J_\varepsilon * f - f\|_p \leq \|J_\varepsilon * f - J_\varepsilon * g\|_p + \|J_\varepsilon * g - g\|_p + \|g - f\|_p
\]
\[
\leq 2\|g - f\|_p + \|J_\varepsilon * g - g\|_p
\]
\[
< 3\delta.
\]
This proves (v).

**Corollary 13.** \( C_c^\infty(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d) \) for \( 1 \leq p < \infty \).

**Proof.** This follows by approximating \( f \in L^p(\mathbb{R}^d) \) with a function \( g \in C_c(\mathbb{R}^d) \) using Lemma 10 and then approximating \( g \) with a function in \( C_c^\infty(\mathbb{R}^d) \) using Theorem 12.
The following construction of a smooth ‘cut-off function’ is often useful.

**Corollary 14.** There exists a function \( \varphi \in C_c^\infty(\mathbb{R}^d) \) with \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) on \( \overline{B_1(0)} \) and \( \text{supp } \varphi \subseteq B_2(0) \).

**Proof.** Take \( f = \chi_{B_{3/2}(0)}(x) \) and

\[
\varphi(x) = (J_{1/2} * f)(x) = \int_{|x-y| \leq 3/2} J_{1/2}(y) \, dy.
\]

Since \( \int_{\mathbb{R}^d} J_{1/2}(y) \, dy = 1 \) and \( J_{1/2} \geq 0 \), it follows that \( 0 \leq \varphi \leq 1 \). Moreover, if \( |x| \leq 1 \), then \( \text{supp } J_{1/2} = \overline{B_{1/2}(0)} \subseteq B_{3/2}(x) \). Hence, \( \varphi(x) = 1 \) on \( \overline{B_{1/2}(0)} \). Finally, \( \text{supp } \varphi \subseteq \text{supp } J_{1/2} + \text{supp } f \subseteq B_2(0) \). \( \square \)

**References**

