Solutions

1. a) We have that
   \[ \alpha = \frac{\sqrt{2}}{2} \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = \frac{\sqrt{2}e^{\pi i/4}}{2e^{\pi i/6}} = e^{\pi i/4} - \frac{\pi i}{6} \sqrt{2} = 2 - \frac{1}{2}e^{\pi i/12}. \]

   b) Set \( z = re^{i\theta} \). The equation can then be written as
   \[ r^4 e^{4i\theta} = 2 - \frac{1}{2} \ e^{\pi i/12} \quad \Rightarrow \quad r = 2^{-1/8} \quad \text{and} \quad \theta = \frac{\pi}{48} + k\frac{\pi}{2}, \ k \in \mathbb{Z}, \]
   and hence
   \[ z = z_k = 2^{-1/8}e^{(\pi/48+k\pi/2)i}, \ k = 0, 1, 2, 3. \]

2. We use induction on \( n \). We have that
   \[ LHS_1 = (-1)^1 \cdot 1^2 = -1 \quad \text{and} \quad RHS_1 = \frac{(-1)^1 \cdot 1 \cdot (1 + 1)}{2} = -1, \]
   which proves the statement for \( n = 1 \).
   Assume that \( LHS_n = RHS_n \) where \( n \) is a positive integer. Then
   \[ LHS_{n+1} = \sum_{k=1}^{n+1} (-1)^k k^2 = \sum_{k=1}^{n} (-1)^k k^2 + (-1)^{n+1}(n + 1)^2 = LHS_n + (-1)^{n+1}(n + 1)^2 \]
   \[ = RHS_n + (-1)^{n+1}(n + 1)^2 = \frac{(-1)^n n(n + 1)}{2} + (-1)^{n+1}(n + 1)^2 \]
   \[ = \frac{(-1)^{n+1}(n + 1)(-n + 2(n + 1))}{2} = \frac{(-1)^{n+1}(n + 1)(n + 2)}{2} = RHS_{n+1}. \]
   This concludes the proof.
3. a) The other rank can be chosen in 12 different ways. The two aces can then be chosen in \( \binom{4}{2} \) ways and the two cards of the other rank can also be chosen in \( \binom{4}{2} \) ways. Hence, there are \( 12 \binom{4}{2}^2 = 432 \) hands of the specified kind.

b) If none of the aces is a club, then the two aces can be chosen in \( \binom{3}{2} \) ways. The other two cards, which then must be clubs and not aces, can be chosen in \( \binom{12}{2} \) ways. If ace of clubs is included, then the other ace can be chosen in 3 ways. We must then choose one more club and a card of another suit. This can be done in \( 12 \cdot 36 \) ways. Now the number of hands of the specified kind is \( \binom{3}{2} \binom{12}{2} + 3 \cdot 12 \cdot 36 = 1494 \).

4. We apply the Euclidean algorithm to the two polynomials,
\[
x^4 + 4x^3 + 3x^2 - 8x - 10 = x^4 + 6x^3 + 8x^2 - 12x - 20 - (2x^3 + 5x^2 - 4x - 10),
\]
\[
2(x^4 + 6x^3 + 8x^2 - 12x - 20) = (x + \frac{7}{2})(2x^3 + 5x^2 - 4x - 10) + \frac{5}{2}(x^2 - 2),
\]
and find that \( x^2 - 2 \) is a greatest common divisor. Hence, the common roots are \( \pm \sqrt{2} \).

Dividing the first polynomial by this greatest common divisor, we obtain the quotient \( x^2 - 2 \) and find that the zeros of the polynomial \( x^2 - 2 \) are \( \pm \sqrt{2} \). Hence, the roots of the first equation are \( \pm \sqrt{2} \) and \( -2 \pm i \).

5. a) We have that
\[
\cos (5\theta) + i \sin (5\theta) = e^{5i\theta} = (e^{i\theta})^5 = (\cos \theta + i \sin \theta)^5 = \sum_{k=0}^{5} \binom{5}{k} (\cos \theta)^{5-k} (i \sin \theta)^k
\]
Identifying imaginary parts, we get that
\[
\sin (5\theta) = 5 \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^3 \theta + 5 \cos \theta \sin^5 \theta
\]
\[
= 5(1 - \sin^2 \theta)^2 \sin \theta - 10(1 - \sin^2 \theta) \sin^3 \theta + 5 \sin \theta
\]
\[
= 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta.
\]

b) For \( \theta = \pm \frac{\pi}{5} \) and \( \theta = \pm \frac{2\pi}{5} \), we have that \( \sin \theta \neq 0 \) and
\[
(\sin \theta)(5 - 20 \sin^2 \theta + 16 \sin^4 \theta) = \sin (5\theta) = 0.
\]
Hence, \( \pm \sin \left( \frac{\pi}{5} \right) \) and \( \pm \sin \left( \frac{2\pi}{5} \right) \) are distinct zeros of the polynomial
\[
f = 5 - 20t^2 + 16t^4.
\]

c) The zeros of the polynomial \( f \) are given by
\[
t^4 - \frac{5}{4}t^2 + \frac{5}{16} = 0 \iff \left( t^2 - \frac{5}{8} \right)^2 - \frac{5}{64} = 0 \iff t^2 = \frac{5 \pm \sqrt{5}}{8}
\]
\[
\iff t = \pm \sqrt{\frac{5 \pm \sqrt{5}}{8}} = \pm \sqrt{\frac{10 \pm 2\sqrt{5}}{4}}.
\]
Since \( \sin \left( \frac{\pi}{5} \right) \) is the least positive zero of the polynomial, we obtain that
\[
\sin \left( \frac{\pi}{5} \right) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.
\]
6. By the definition of Fibonacci numbers, we have that

\[ f_{n+3} = f_{n+2} + f_{n+1} = f_{n+1} + f_n + f_n = 2f_{n+1} + f_n \equiv f_n \pmod{2}. \]

It follows from this, that

\[ f_{n+60} \equiv f_{n+57} \equiv f_{n+54} \equiv \cdots \equiv f_n \pmod{2}. \]

We also get that

\[ f_{n+5} = 2f_{n+3} + f_{n+2} = 2(2f_{n+1} + f_n) + f_{n+1} + f_n = 5f_{n+1} + 3f_n \equiv 3f_n \pmod{5}. \]

Hence, by Fermat’s little theorem,

\[ f_{n+60} \equiv 3^{12}f_n \equiv (3^4)^3f_n \equiv f_n \pmod{5}. \]

Since 2 and 5 are different primes, we get that \( f_{n+60} \equiv f_n \pmod{10}, \) from which the assertion follows.