Solutions

1. a) The series
\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{\ln k} \]
is **convergent**, since it satisfies the conditions of Leibniz’ criterion for convergence:
- The terms are of **alternate sign**.
- The terms tend to zero as \( k \) tends to \( \infty \).
- The absolute value of the terms is a **decreasing function** of \( k \) because the denominator \( \ln k \) is an increasing function of \( k \).

b) The series
\[ \sum_{k=1}^{\infty} \frac{2^k k!}{k^k} \]
is **convergent**. With \( a_k = \frac{2^k k!}{k^k} \) we have
\[
\frac{a_{k+1}}{a_k} = \frac{2^{k+1} (k+1)!}{(k+1)^{k+1}} = \frac{2(1 + 1/k)^{-k}}{e} \to \frac{2}{e} \quad \text{as} \quad k \to \infty.
\]
The convergence now follows from the ratio test since the limit is less than 1.

c) The series
\[ \sum_{k=1}^{\infty} \frac{(2 + 3i)^k}{(3 + 2i)^k} \]
is **divergent** since the absolute value of the term is
\[
\left| \frac{2 + 3i}{3 + 2i} \right| = 1
\]
so the terms do not tend to zero.

2. Let \( v(x,t) = u(x,t) - x \). Then \( v \) solves the problem
\[
\begin{cases}
\partial_t v(x,t) = 3 \partial_{xx}^2 v(x,t), & 0 < x < \pi, \quad t > 0, \\
\partial_x v(0,t) = \partial_x v(\pi,t) = 0, & t > 0, \\
v(x,0) = \cos 4x \cos 2x, & 0 < x < \pi.
\end{cases}
\]
By Euler’s formulas,
\[
\cos 4x \cos 2x = \left( \frac{e^{4ix} + e^{-4ix}}{2} \right) \left( \frac{e^{2ix} + e^{-2ix}}{2} \right) = \frac{\cos 2x}{2} + \frac{\cos 6x}{2}.
\]

Please, turn over!
Hence the solution is

\[ v(x, t) = \frac{1}{2}(e^{-12t} \cos 2x + e^{-108t} \cos 6x). \]

and

\[ u(x, t) = x + \frac{1}{2}(e^{-12t} \cos 2x + e^{-108t} \cos 6x). \]

3. a) By using Euler’s formula for \( \sin x \), we find that

\[
\begin{align*}
    c_n &= \frac{1}{4\pi i} \int_{-\pi}^{\pi} e^{x(1+i-in)} - e^{x(1-i-in)} \, dx \\
    &= \frac{1}{4\pi i} \left[ \frac{e^{x(1+i-in)}}{1+i(1-n)} - \frac{e^{x(1-i-in)}}{1-i(1+n)} \right]_{-\pi}^{\pi}
\end{align*}
\]

Inserting \( \pi \) and \( -\pi \), and using that \( e^{ik \pi} = (-1)^k \) for any integer \( k \) and \( \sinh x = (e^x - e^{-x})/2 \) gives

\[
    c_n = \frac{(-1)^{n+1}}{2\pi i} \frac{\sinh \pi}{1+i(1-n)} - \frac{1}{1-i(1+n)} = \frac{(-1)^{n+1} \sinh \pi}{\pi} \frac{1}{n^2 + 2in - 2}
\]

Thus the Fourier series of \( u \) is

\[
u(x) = \frac{\sinh \pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^{n+1} e^{inx}}{n^2 + 2in - 2}
\]

The function \( u \) is equal to the sum of its Fourier series, since it is continuous and piecewise \( C^1 \).

b) The sum of the series for \( x = 3\pi/2 \) is \( u(3\pi/2) \). Since \( u \) has period \( 2\pi \),

\[
u(3\pi/2) = u(-\pi/2) = e^{-\pi/2} \sin(-\pi/2) = -e^{-\pi/2}.
\]

c) First note that

\[
\frac{1}{n^2 + 2in - 2} = \frac{n^2 - 2in - 2}{(n^2 - 2)^2 + 4n^2} = \frac{n^2 - 2}{n^4 + 4} + i \frac{-2n}{n^4 + 4}
\]

The terms of the series in question are thus the real parts of the Fourier coefficients. For \( x = \pi \), we have

\[
u(\pi) = 0 = -\frac{\sinh \pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{n^2 - 2}{n^4 + 4} + i \frac{-2n}{n^4 + 4}
\]

The sum of the real parts is zero, so

\[
0 = -\frac{1}{2} + 2 \sum_{k=1}^{\infty} \frac{n^2 - 2}{n^4 + 4}
\]

It follows that

\[
\sum_{n=1}^{\infty} \frac{n^2 - 2}{n^4 + 4} = \frac{1}{4}
\]
4. Assume that \( u \) is given by a power series with a positive radius of convergence, \( u(x) = \sum_{k=0}^{\infty} a_k x^k \). We can differentiate term by term if \( x \) is within the radius of convergence:

\[
\begin{align*}
    u'(x) &= \sum_{k=1}^{\infty} k a_k x^{k-1}, \\
    u''(x) &= \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}
\end{align*}
\]

After insertion in the differential equation we get

\[
\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.
\]

We replace \( k - 2 \) by \( k \) in the first series:

\[
\sum_{k=0}^{\infty} (k + 2)(k + 1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.
\]

If the equality is valid for all \( x \) in a neighborhood of 0 then the coefficient for every power of \( x \) is zero. It follows that

\[
(k + 2) a_{k+2} + a_k = 0, \quad k \geq 0.
\]

The initial values imply that \( a_0 = 1 \) and \( a_1 = 0 \). Hence all coefficients with odd indices are zero and

\[
a_{2k} = \frac{(-1)^k}{2^k k!}.
\]

The solution is

\[
u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} = e^{-x^2/2}.
\]

5. a) For \( x \geq 0 \) we have

\[
0 \leq \frac{x}{2 + k^3 x} \leq \frac{1}{k^3}
\]

so the series is uniformly convergent for \( x \geq 0 \) by Weierstrass’ M-test since \( \sum k^{-3} \) is convergent. It follows that \( f \) is continuous for \( x \geq 0 \).

b) The derivative of term number \( k \) is

\[
g_k(x) = \frac{2}{(2 + k^3 x)^2}
\]

Let \( a > 0 \). The supremum of \( g_k(x) \) for \( x \geq a \) is less than or equal to \( \frac{2}{(2 + ak^3)^2} \) and the series

\[
\sum_{k=1}^{\infty} \frac{2}{(2 + ak^3)^2}
\]

is convergent since \( a \neq 0 \). By Weierstrass’ M-test, the differentiated series is uniformly convergent for \( x \geq a \). Therefore, the function \( f \) is differentiable for \( x \geq a \). Since this is true for any \( a > 0 \), it follows that \( f \) is differentiable for \( x > 0 \).

c) We have \( f(0) = 0 \). If \( f \) has a right derivative at 0, it is equal to

\[
\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \sum_{k=1}^{\infty} \frac{1}{(2 + k^3 x)^2}
\]

Let \( N \) be any positive integer. Since the terms of the series are positive, the limit is greater than the limit of the first \( N \) terms, which is equal to \( N/4 \). Since this is true for any \( N \), the limit is \( \infty \) and the derivative does not exist.