Solutions

1. Set

\[ A = \begin{bmatrix} 1 & 2 & -3 & -7 \\ 2 & 5 & -7 & -15 \end{bmatrix}. \]

Then \( x \in \ker A \) if and only if

\[
A x = 0 \iff \begin{bmatrix} 1 & 2 & -3 & -7 \\ 2 & 5 & -7 & -15 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 & -3 & -7 \\ 0 & 1 & -1 & -1 \end{bmatrix} \]

for some real numbers \( s \) and \( t \). Hence, the vectors \( v_1 = (1, 1, 1, 0) \) and \( v_2 = (5, 1, 0, 1) \) span \( \ker A \), and since they are linearly independent, they form a basis for \( \ker A \). We now apply the Gram–Schmidt process to them to obtain an orthonormal basis for \( \ker A \).

Set \( u_1 = v_1 \) and \( u_2 = s u_1 + v_2 \). Then

\[
\langle u_1, u_2 \rangle = 0 \iff s = -\frac{\langle u_1, v_2 \rangle}{\|u_1\|^2} = -\frac{6}{3} = -2.
\]

Hence, \( u_2 = -2u_1 + v_2 = (3, -1, -2, 1) \) is orthogonal to \( u_1 \) and

\[
e_1 = \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{3}}(1, 1, 1, 0) \quad \text{and} \quad e_2 = \frac{1}{\|u_2\|} u_2 = \frac{1}{\sqrt{15}}(3, -1, -2, 1)
\]

form an orthonormal basis for \( \ker A \).

**Answer:** E.g. \( \frac{1}{\sqrt{3}}(1, 1, 1, 0), \frac{1}{\sqrt{15}}(3, -1, -2, 1) \).

2. The eigenvalues of the coefficient matrix are given by

\[
0 = \begin{vmatrix} 1 - \lambda & 2 & 1 \\ 2 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 1 + \lambda & 1 \\ 2 & -1 - \lambda & -1 \\ -1 & 0 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda & 1 + \lambda & 1 \\ 3 - \lambda & 0 & 0 \\ -1 & 0 & 1 - \lambda \end{vmatrix} = -(\lambda + 1)(\lambda - 1)(\lambda - 3)
\]

and are \( \lambda_1 = -1, \lambda_2 = 1 \) and \( \lambda_3 = 3 \).

The eigenvectors belonging to \( \lambda_1 \) are the non-zero vectors given by

\[
\begin{bmatrix} 2 & 2 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \iff \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \iff x = te_1
\]

where \( e_1 = (1, -1, 0) \).
The eigenvectors belonging to $\lambda_2$ are the non-zero vectors given by

$$
\begin{bmatrix}
0 & 2 & 1 \\
2 & 0 & -1 \\
-1 & -1 & 0
\end{bmatrix}
\equiv
\begin{bmatrix}
0 & 2 & 1 \\
2 & 2 & 0 \\
-1 & -1 & 0
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
= te_2
$$

where $e_2 = (1, -1, 2)$.

Those belonging to $\lambda_3$ are the non-zero vectors given by

$$
\begin{bmatrix}
-2 & 2 & 1 \\
-2 & -2 & -1 \\
-1 & -1 & -2
\end{bmatrix}
\equiv
\begin{bmatrix}
-4 & 0 & -3 \\
-1 & -1 & -2 \\
-1 & -1 & -2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
= te_3
$$

where $e_3 = (3, 5, -4)$.

Since the eigenvalues are distinct, the eigenvectors $e_1, e_2, e_3$ form a basis for $\mathbb{R}^3$. The general solution of the system of linear equations is therefore given by

$$
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
= c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}.
$$

The initial condition gives

$$
\begin{bmatrix}
1 & 1 & 3 \\
-1 & -1 & 5 \\
0 & 2 & -4
\end{bmatrix}
\equiv
\begin{bmatrix}
1 & 1 & 3 \\
0 & 0 & 8 \\
0 & 2 & -4
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= \begin{bmatrix}
-3 \\
-2 \\
2
\end{bmatrix}.
$$

Answer: $\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
= -3e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2e^{t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$.

3. The matrix of the quadratic form $q$ is

$$
B = \begin{bmatrix}
6 & -2 & 2 \\
-2 & -1 & 0 \\
2 & 0 & -1
\end{bmatrix}
$$

and since

$$
\det (B - \lambda I) = \begin{vmatrix}
6 - \lambda & -2 & 2 \\
-2 & -1 - \lambda & 0 \\
2 & 0 & -1 - \lambda
\end{vmatrix} = \begin{vmatrix}
6 - \lambda & -2 & 0 \\
-2 & -1 - \lambda & -1 - \lambda \\
2 & 0 & -1 - \lambda
\end{vmatrix} = -(\lambda + 2)(\lambda + 1)(\lambda - 7),
$$

the eigenvalues of $q$ are $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 7$. Two eigenvalues are negative, one is positive and the right-hand side of the equation is positive. The surface is therefore a hyperboloid of two sheets. Let $e'_1, e'_2, e'_3$ be an orthonormal basis of eigenvectors belonging to $\lambda_1, \lambda_2, \lambda_3$. Then

$$
q(x) = -2(x'_1)^2 - (x'_2)^2 + 7(x'_3)^2
$$
if \( x = x'_1 e'_1 + x'_2 e'_2 + x'_3 e'_3 \). For a point \( x \) on the surface, we have that

\[
\|x\|^2 = \|x'\|^2 = \frac{1}{7}(7(x'_1)^2 + 7(x'_2)^2 + 7(x'_3)^2) \geq \frac{1}{7}(-2(x'_1)^2 - (x'_2)^2 + 7(x'_3)^2) = \frac{28}{7} = 4
\]

with equality if and only if \( x'_1 = x'_2 = 0 \) and \( x'_3 = \pm 2 \). Since

\[
Bx = \lambda_3 x \iff \begin{bmatrix}
-1 & -2 & 2 \\
2 & -8 & 0 \\
0 & 0 & 0
\end{bmatrix} \iff \begin{bmatrix}
-1 & -2 & 2 \\
0 & -8 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

we can take \( e'_3 = \frac{1}{\sqrt{18}}(4, -1, 1) \). The points on the surface closest to the origin are therefore

\[
\pm 2 e'_3 = \pm \frac{\sqrt{2}}{3}(4, -1, 1).
\]

**Answer:** The surface is a hyperboloid of two sheets and the points on the surface closest to the origin are \( \pm \frac{\sqrt{2}}{3}(4, -1, 1) \).

4. If \( A \) is the matrix of \( F \), we have that

\[
\langle A_1, A_1 \rangle = \frac{1}{49}(36 + 4 + 9) = 1,
\]

\[
\langle A_2, A_2 \rangle = \frac{1}{49}(4 + 9 + 36) = 1,
\]

\[
\langle A_3, A_3 \rangle = \frac{1}{49}(9 + 36 + 4) = 1,
\]

\[
\langle A_1, A_2 \rangle = \frac{1}{49}(12 + 6 - 18) = 0,
\]

\[
\langle A_1, A_3 \rangle = \frac{1}{49}(-18 + 12 + 6) = 0,
\]

\[
\langle A_2, A_3 \rangle = \frac{1}{49}(-6 + 18 - 12) = 0.
\]

Hence, \( A \) is orthogonal, and since the basis is orthonormal, \( F \) is an isometry. From

\[
A x = x \iff \begin{bmatrix}
-1 & 2 & -3 \\
2 & -4 & 6 \\
3 & -6 & -5
\end{bmatrix} \iff \begin{bmatrix}
-1 & 2 & -3 \\
0 & -4 & 15
\end{bmatrix} \iff x = t(2, 1, 0),
\]

we conclude that \( \ker (A - I) \) is the line spanned by the vector \( w \) with coordinates \((2, 1, 0)\). This shows that \( F \) is a rotation about that line. Take \( u \) to be the vector with coordinates \((1, -2, 0)\). Then \( u \) is orthogonal to \( w \). The coordinates of \( v = F(u) \) are

\[
A \begin{bmatrix}
1 \\
-2 \\
0
\end{bmatrix} = \frac{1}{7} \begin{bmatrix}
2 \\
-4 \\
15
\end{bmatrix}
\]

The angle of rotation \( \theta \) is the angle between \( u \) and \( v \), and we have that

\[
\cos \theta = \frac{\langle u, v \rangle}{\|u\|\|v\|} = \frac{10}{\sqrt{245} \sqrt{3}} = \frac{10}{35} = \frac{2}{7}.
\]

*Please, turn over!*
The determinant of the matrix whose columns are the coordinates of \( \mathbf{u}, 7\mathbf{v} \) and \( \mathbf{w} \) is

\[
\begin{vmatrix}
1 & 2 & 2 \\
-2 & -4 & 1 \\
0 & 15 & 0
\end{vmatrix}
= -15 \begin{vmatrix}
1 & 2 \\
-2 & 1
\end{vmatrix}
= -75 < 0.
\]

This shows that the rotation appears to be clockwise when looking from the point with coordinates \((2,1,0)\) towards the origin.

**Answer:** \( F \) is rotation about the line spanned by the vector with coordinates \((2,1,0)\) through the angle \( \theta \) where \( \cos \theta = \frac{7}{15} \) in the clockwise direction when looking from the point with coordinates \((2,1,0)\) towards the origin.

5. See the proof of Theorem 3.29.

6. a) We have that

\[
\begin{align*}
\mathbf{x} & \in V_\lambda \quad \Rightarrow \quad A\mathbf{x} = \lambda \mathbf{x} \\
\Rightarrow \quad A(B\mathbf{x}) &= (AB)\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B(\lambda \mathbf{x}) = \lambda (B\mathbf{x}) \\
\Rightarrow \quad B\mathbf{x} & \in V_\lambda.
\end{align*}
\]

This proves the assertion.

b) By the spectral theorem, \( \mathbb{R}^n \) has a basis consisting of eigenvectors of \( A \). Let \( \lambda_1, \ldots, \lambda_k \) be the distinct eigenvalues of \( A \). Then every vector \( \mathbf{u} \in \mathbb{R}^n \) can be written uniquely as

\[
\mathbf{u} = \mathbf{u}_1 + \cdots + \mathbf{u}_k
\]

where \( \mathbf{u}_i \in V_{\lambda_i} \) for \( i = 1, \ldots, k \). By the result shown in 6 a), the restriction of the mapping \( \mathbf{x} \mapsto B\mathbf{x} \) to \( V_{\lambda_i} \) is a symmetric linear transformation on \( V_{\lambda_i} \). Hence, each \( V_{\lambda_i} \) has a basis consisting of eigenvectors of \( B \). These vectors are also eigenvectors of \( A \), and \( \mathbf{u}_i \) can be written uniquely as a linear combination of them. Consequently, \( \mathbf{u} \) can be written uniquely as a linear combination of all these eigenvectors taken together, which therefore form a basis for \( \mathbb{R}^n \).