1. Form the Lagrange function \( L(x, y, \lambda) = x + 8y - \lambda(x^4 + y^4 - 17) \) and look for critical points. We obtain the system

\[
\begin{align*}
L'_x &= 1 - 4\lambda x^3 = 0 \quad (0.1) \\
L'_y &= 8 - 4\lambda y^3 = 0 \quad (0.2) \\
L'_\lambda &= -(x^4 + y^4 - 17) = 0. \quad (0.3)
\end{align*}
\]

The first two conditions imply that \( 4\lambda = 1/x^3 = 8/y^3 \), so \( y = 2x \) at a critical point. Inserting this in the last condition gives \( 17x^4 = 17 \) or \( x = \pm 1 \). This gives us two candidates for extreme points: \((x, y) = \pm(1, 2)\). The corresponding values are calculated to \( f(1, 2) = 17 \) and \( f(-1, -2) = -17 \). Since the gradient of the function \( g(x, y) = x^4 + y^4 - 17 \) equals \( 4(x^3, y^3) \) which is not the zero-vector when \( x^4 + y^4 = 17 \) there are no other candidates for extreme values, so the minimum is \(-17\) and the maximum is \(17\).

2. By Green’s formula, the integral equals to

\[
\iint_D (2y+1) \, dx \, dy = \int_0^1 dx \int_x^1 (2y+1) \, dy = \int_0^1 (x^2 - x^4 + (x-x^2)) \, dx = 3/10.
\]

3. By the divergence theorem, the flux in question can be written \( I_1 + I_2 \) where

\[
I_1 = \iiint_{x^2+y^2+z^2\leq a^2, z\geq 0} (x^2 + y^2 + z^2) \, dV,
\]

\[
I_2 = \iint_{x^2+y^2\leq a^2} e^{x^2+y^2} \, dA.
\]

Passing to spherical coordinates, we see that

\[
I_1 = \int_{r=a}^{\pi/2} \int_0^{2\pi} \int_0^a r^2 \sin \phi \, dr \, d\theta \, d\phi \sin \phi \, d\phi = \int_0^a r^4 \, dr \int_0^{2\pi} \int_0^{\pi/2} \sin \phi \, d\phi \, d\phi = \frac{a^5}{5} \cdot 2\pi \cdot 1,
\]

and similarly

\[
I_2 = \int_0^{2\pi} d\theta \int_0^a r e^{-r^2} \, dr = 2\pi \cdot \frac{1}{2} (e^{a^2} - 1).
\]

The answer is thus

\[
2\pi \frac{a^5}{5} + \pi (e^{a^2} - 1).
\]
4. It is readily seen that \( g_n(x) \to 0 \) as \( n \to \infty \) for all \( x \geq 0 \). We have
\[
g'_n(x) = n^\alpha e^{-nx}(1 - nx).
\]
From this, and the observation that \( g_n(0) = \lim_{x \to \infty} g_n(x) = 0 \), we see that \( g_n \) assumes its maximum on \([0, \infty)\) at the point \( x = 1/n \), and
\[
g_n(1/n) = n^{\alpha-1}/e.
\]
This converges to 0 if and only if \( \alpha < 1 \). We have shown that the convergence is uniform precisely when \( \alpha < 1 \).

5. Since we are dealing with a field defined on \( \mathbb{R}^3 \) (a simply connected set), the field is conservative if (and only if) it is irrotational. A calculation gives that
\[
\text{curl } u = (0, f(x) \cos y - f'(x) \cos y, -f'(x) \sin y + f(x) z \sin y).
\]
This vanishes if \( f' = f \), i.e. if \( f(x) = C e^x \). Hence
\[
u = C(e^x z \cos y, -e^x z \sin y, e^x \cos y).
\]
The condition \( u(0,0,0) = (0,0,1) \) now gives \( C = 1 \). Finally, a potential function is calculated to \( \phi = e^x \cos(y)z \).

6. By the triangle inequality for integrals, we have
\[
\left| \frac{1}{2\pi r} \int_{\gamma_r} u \, ds - u(0,0) \right| = \frac{1}{2\pi r} \left| \int_{\gamma_r} (u(x,y) - u(0,0)) \, ds \right|
\leq \frac{1}{2\pi r} \int_{\gamma_r} |u(x,y) - u(0,0)| \, ds.
\]
Now fix \( \varepsilon > 0 \) and take \( \delta > 0 \) small enough that \( |u(x,y) - u(0,0)| < \varepsilon \) when \( x^2 + y^2 < \delta^2 \). The inequality above insures that \( |I(r) - u(0,0)| < \varepsilon \) when \( r < \delta \), proving that \( I(r) \to u(0,0) \) as \( r \to 0 \). This settles part (a).

To prove (b) we use the parametrization \( \gamma_r = r(\cos \theta, \sin \theta), 0 \leq \theta \leq 2\pi \) to obtain \( ds = r \, d\theta \) and
\[
I(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r \cos \theta, r \sin \theta) \, d\theta.
\]
(0.4)

Differentiation under the integral sign gives
\[
I'(r) = \frac{1}{2\pi} \int_0^{2\pi} \left( u_1'(r \cos \theta, r \sin \theta) \cdot \cos \theta + u_2'(r \cos \theta, r \sin \theta) \cdot \sin \theta \right) \, d\theta
\]
\[
= \frac{1}{2\pi r} \int_{\gamma_r} (\nabla u \cdot n) \, ds
\]
where \( \mathbf{n} \) is the unit normal to \( \gamma_r \) which points out from the disk \( D_r : x^2 + y^2 \leq r^2 \). By the divergence theorem we hence have

\[
I'(r) = \frac{1}{2\pi r} \iint_{D_r} \text{div} \nabla u \, dA = \frac{1}{2\pi r} \iint_{D_r} \Delta u \, dA \geq 0,
\]

since we assumed that \( \Delta u \geq 0 \). This shows that the function \( I(r) \) is increasing for \( r \geq 0 \).