Note: Only students who are registered or re-registered on the course are allowed to take the exam.

No aids allowed except for the list of formulas. Use the distributed paper sheets and write only on one side. Fill in the cover sheet completely and write your initials on each sheet. Write legibly (in Swedish or English). Motivate your conclusions clearly and concisely; draw a picture if appropriate.

Test results: Posted Thursday, Nov 6, before 17.00.

Oral exams: Monday, Nov 10 – Thursday, Nov 13. State your preference (day and AM/PM) on the cover sheet of your test — at least two options.

1. Solve the initial value problem

   \[ x' = \frac{tx^2}{x}, \quad x(0) = 1. \]

   What is the maximal interval of definition of the solution?

2. Find \( a \) such that the equation

   \( (x^2 + y^2 + axy) \, dx + (x^2 + axy) \, dy = 0 \)

   is exact. Solve the equation for this \( a \).

3. Find a solution of the problem

   \[ x' + (\cosh t) \ast x = e^t, \quad x(0) = 1, \]

   where

   \( (f \ast g)(t) = \int_0^t f(t - \theta)g(\theta) \, d\theta. \)

4. Solve the initial value problem \( \ddot{x} = Ax, \quad \ddot{x}(0) = \bar{x}_0, \) where

   \[ A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 3 & -3 & 2 \end{pmatrix} \quad \text{and} \quad \bar{x}_0 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \]

5. Show that \( t = 0 \) is a regular singular point of the equation

   \[ tx'' - x' + 4t^3x = 0, \quad t > 0, \]

   and find all possible Frobenius series solutions. Justify the convergence of the series.

Please, turn over!
6. a) Let $B(t) \in \mathbb{R}^{n \times n}$ be continuously differentiable on the interval $I$ and assume that $B(t)$ and $B'(t)$ commute for all $t \in I$. Prove that
\[
\frac{d}{dt} e^{B(t)} = B'(t) e^{B(t)}, \quad t \in I.
\]

b) Let $A(t) \in \mathbb{R}^{n \times n}$ be continuous on the interval $I$ and let $t_0 \in I$. Assume that $A(t)$ and $A(s)$ commute for all $t, s \in I$. Show that the solution of the initial value problem
\[
\bar{x}' = A(t) \bar{x}, \quad \bar{x}(t_0) = \bar{x}_0
\]
on $I$ is given by
\[
\bar{x}(t) = e^{\int_{t_0}^{t} A(s) \, ds} \bar{x}_0.
\]

c) Show by a counterexample that the result in b) is not true if $A(t)$ and $A(s)$ don’t commute for all $t, s \in I$. 