Solutions

1. The equation is separable. We have

\[ x' = te^x \iff e^{-x}x' = t \iff \int e^{-x} \, dx = \int t \, dt. \iff -e^{-x} = \frac{t^2}{2} + C. \]

Using the initial condition we see that \( C = -e^{-2} \). Thus,

\[ e^{-x} = e^{-2} - \frac{t^2}{2} \iff x(t) = -\log \left( e^{-2} - \frac{t^2}{2} \right). \]

The solution is defined as long as \( t^2 \leq 2e^{-2} \), so the maximal interval of existence is \( \left( -\sqrt{2e^{-2}}, \sqrt{2e^{-2}} \right) \).

2. a) \( L[te^{-2t} + 2](s) = L[t](s+2) + 2L[1](s) = \frac{1}{(s+2)^2} + \frac{2}{s} \).

b) We have \( L[x'+x] = sX(s) - x(0) + X(s) = (s+1)X(s) - 1 \), where \( X(s) = L[x](s) \).

Hence,

\[ X(s) = \frac{1}{s+1} + \frac{1}{(s+1)(s+2)^2} + \frac{2}{s(s+1)} \]
\[ = \frac{1}{s+1} + \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} + \frac{2}{s} - \frac{2}{s+1} \]
\[ = -\frac{1}{s+2} - \frac{1}{(s+2)^2} + \frac{2}{s} \]

and

\[ x(t) = -e^{-2t} - te^{-2t} + 2. \]

3. Let

\[ A = \begin{pmatrix} 3 & 5 \\ -5 & -7 \end{pmatrix}. \]

Then \( K_A = \det(\lambda I - A) = (\lambda - 3)(\lambda + 7) + 25 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 \). \( A \) has the only eigenvalue \( \lambda = -2 \). The corresponding eigenspace is one-dimensional,

\[ \mathcal{E}(A, -2) = \text{span}\{(1, -1)\}, \]

and hence there’s no basis consisting of eigenvectors. On the other hand \( \mathcal{G}\mathcal{E}(A, -2) = \mathbb{C}^2 = \mathcal{N}((A + 2I)^2) \), so that \((A + 2I)^2 = 0\). This means that

\[ e^{tA} = e^{-2t}e^{t(A+2I)} = e^{-2t}(I + t(A + 2I)) = e^{-2t} \begin{pmatrix} 1 + 5t & 5t \\ -5t & 1 - 5t \end{pmatrix}. \]

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4. The second equation is \( x''_2 = x_2 \) and has the general solution \( x_2(t) = C_1 e^t \). Substituting this in the first equation, we obtain that
\[
\begin{align*}
x'_1 &= -\frac{1}{t} x_1 + \frac{1}{t} x_2 = -\frac{1}{t} x_1 + \frac{1}{t} C_1 e^t \\
⇔ tx'_1 + x_1 &= C_1 e^t \\
⇔ (tx_1)' &= C_1 e^t \\
⇔ tx_1 &= C_1 e^t + C_2 \\
⇔ x_1 &= C_1 \frac{e^t}{t} + C_2 \frac{1}{t},
\end{align*}
\]

Hence, the general solution is
\[
x(t) = C_1 \left( \frac{x_1'}{e^t} \right) + C_2 \left( \frac{1}{t} \right), \quad t > 0.
\]

5. The equation can be rewritten in the form \( x'' + P(t)x' + Q(t)x = 0 \), where
\[
tP(t) = \frac{3}{2} \quad \text{and} \quad t^2 Q(t) = -\frac{1 + t}{2}
\]
are analytic at \( t = 0 \). Hence, \( t = 0 \) is a regular singular point. The indicial equation is \( m(m-1) + \frac{3}{2} m - \frac{1}{2} = 0 \) or \( (m+1)(m-\frac{1}{2}) = 0 \) with roots \( m_1 = \frac{1}{2} \) and \( m_2 = -1 \). Since \( m_1 - m_2 = \frac{3}{2} \notin \mathbb{Z} \), there are two linearly independent Frobenius series solutions.

Making the Ansatz \( x(t) = \sum_{n=0}^{\infty} a_n t^{n+m}, \ a_0 \neq 0 \), we find that
\[
\begin{align*}
x'(t) &= \sum_{n=0}^{\infty} (n + m) a_n t^{n+m-1}, \\
x''(t) &= \sum_{n=0}^{\infty} (n + m)(n + m - 1) a_n t^{n+m-2}, \\
\frac{3}{2t} x'(t) &= \sum_{n=0}^{\infty} \frac{3}{2} (n + m) a_n t^{n+m-2}, \\
-\frac{1}{2t^2} x(t) &= -\sum_{n=0}^{\infty} \frac{1}{2} a_n t^{n+m-2}, \\
-\frac{1}{2t} x(t) &= -\sum_{n=1}^{\infty} \frac{1}{2} a_{n-1} t^{n+m-2}.
\end{align*}
\]

Hence,
\[
\left[ m(m-1) + \frac{3}{2} m - \frac{1}{2} \right] a_0 = 0,
\]
and
\[ \left( m + n \right) \left( m + n - 1 \right) + \frac{3}{2} \left( m + n \right) - \frac{1}{2} \right] a_n = \frac{1}{2} a_{n-1}, \quad \text{for } n \geq 2. \]

The first equation is simply the indicial equation. The second equation can be simplified to
\[ \left( n + m + 1 \right) \left( n + m - \frac{1}{2} \right) a_n = \frac{1}{2} a_{n-1}, \quad \text{for } n \geq 1. \]

For \( m_1 = \frac{1}{2} \) we obtain for \( n \geq 1 \) that
\[ \left( n + \frac{3}{2} \right) a_n = \frac{1}{2} a_{n-1}, \]
\[ \iff \quad \left( 2n + 3 \right) a_n = a_{n-1}. \]

Taking \( a_0 = 1 \), we find that
\[ a_n = \frac{a_{n-1}}{\left( 2n + 3 \right) n} = \cdots = \frac{1}{\left( 2n + 3 \right) \left( 2n + 1 \right) \cdots 5 \cdot n!}, \quad n \geq 1. \]

Hence
\[ x_1(t) = t^2 \sum_{n=0}^{\infty} \frac{1}{\left( 2n + 3 \right) \left( 2n + 1 \right) \cdots 5 \cdot n!} t^n, \quad t > 0, \]
is a solution. For \( n = 0 \) one should interpret \( \left( 2n + 3 \right) \left( 2n + 1 \right) \cdots 5 \) as an empty product with value 1. The fact that the series converges for all \( t \) follows from Theorem A in Section 29 of Simmons (since the power series expansions of \( tP(t) \) and \( t^2Q(t) \) both have infinite radius of convergence) or by direct use of the ratio test.

For \( m_2 = -1 \), we obtain for \( n \geq 1 \) that
\[ n \left( n - \frac{3}{2} \right) a_n = \frac{1}{2} a_{n-1} \]
\[ \iff \quad n \left( 2n - 3 \right) a_n = a_{n-1}. \]

Choosing \( a_0 = 1 \), we obtain that
\[ a_n = \frac{a_{n-1}}{\left( 2n - 3 \right) n} = \cdots = \frac{1}{\left( 2n - 3 \right) \left( 2n - 5 \right) \cdots (-1) \cdot n!}, \quad n \geq 1. \]

Hence
\[ x_2(t) = \frac{1}{t} \sum_{n=0}^{\infty} \frac{1}{\left( 2n - 3 \right) \left( 2n - 5 \right) \cdots (-1) \cdot n!} t^n, \quad t > 0, \]
is also a solution. The series converges for all \( t \) by the same reasons as above. It is clear that \( x_1 \) and \( x_2 \) are linearly independent since \( \lim_{t \to 0^+} x_1(t) = 0 \), while \( \lim_{t \to 0^+} x_2(t) = \infty \). Hence, the general solution is \( x(t) = C_1 x_1(t) + C_2 x_2(t) \), where \( C_1 \) and \( C_2 \) are arbitrary constants.

6. Suppose first that \( x_0 > 0 \). Then \( x(t) > 0 \) in a neighborhood of \( t = 0 \). Since the equation is separable, we obtain that
\[ \int_{x_0}^{x(t)} \frac{dx}{\sqrt{x}} = -\frac{t^2}{2} \quad \iff \quad 2\sqrt{x(t)} = 2\sqrt{x_0} - \frac{t^2}{2} \quad \iff \quad x(t) = \left( \sqrt{x_0} - \frac{t^2}{4} \right)^2, \]
as long as \( x(t) > 0 \). Set \( T = 2x_0^{1/4} \). At \( t = \pm T \), we obtain by continuity that \( x(0) = 0 \).

We can extend the solution to the whole real line by setting
\[ x(t) = \begin{cases} \left( \sqrt{x_0} - \frac{t^2}{4} \right)^2, & |t| \leq T \\ 0, & |t| > T. \end{cases} \]

Please, turn over!
Indeed, this defines a continuously differentiable function since
\[
\left(\sqrt{x_0} - \frac{t^2}{4}\right)^2 = \frac{1}{16}(t - T)^2(t + T)^2
\]
vanishes to second order at \( t = \pm T \). It also clearly satisfies the differential equation.

In order to show that there is no other solution, we must prove that \( x(t) \neq 0 \) for \( t > |T| \)
is impossible. Assume that \( x(t) > 0 \) for some \( t > T \). Then \( x'(\tau) > 0 \) and \( x(\tau) > 0 \) for
some \( \tau \in (T, t) \) by the mean value theorem. But then \( x'(\tau) = -\tau \sqrt{|x(\tau)|} < 0 \). This
is a contradiction, so we must have \( x(t) \leq 0 \) for all \( t > T \). A similar argument shows
that \( x(t) < 0 \) is not possible for \( t > T \), since the right hand side of the differential
equation is positive when \( t > 0 \) and \( x < 0 \). Similarly, we must have \( x(t) = 0 \) for
\( t < -T \) (here one can use the fact that \( x(-t) \) is a solution if \( x(t) \) is a solution).

If \( x_0 < 0 \), we can consider the function \( -x(t) \), which is a solution of the equation with
initial value \( -x_0 > 0 \). For \( x_0 = 0 \) the solution must be identically 0 by the above
argument.

All solutions intersect since they are 0 for large enough values of \( |t| \).
This is possible since the right hand side does not satisfy a Lipschitz condition in \( x \)
in a neighborhood of \( x = 0 \).