Solutions

1. The equation is separable. We have

\[ x' = (x^2 + 1)t \leftrightarrow \frac{x'}{x^2 + 1} = t \leftrightarrow \int \frac{1}{x^2 + 1} \, dx = \int t \, dt \leftrightarrow \arctan x(t) = \frac{t^2}{2} + C. \]

The initial condition gives \( C = \frac{\pi}{4} \). Thus,

\[ \arctan x(t) = \frac{t^2}{2} + \frac{\pi}{4} \leftrightarrow x(t) = \tan \left( \frac{t^2}{2} + \frac{\pi}{4} \right). \]

The solution is defined as long as \( -\frac{\pi}{2} < \frac{t^2}{2} + \frac{\pi}{4} < \frac{\pi}{2} \) \( \Leftrightarrow \) \( t^2 < \frac{\pi}{2} \). Hence the maximal interval of definition is \(( -\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}} )\).

2. Let \( M(x, y) = y^3 - 2y \cos x \) and \( N(x, y) = a \sin x + bxy^2 \). Then

\[ \frac{\partial M(x, y)}{\partial y} = 3y^2 - 2 \cos x, \quad \frac{\partial N(x, y)}{\partial x} = a \cos x + by^2, \]

so the equation is exact precisely when \( a = -2 \) and \( b = 3 \). With \( a = -2 \) and \( b = 3 \) we can e.g. compute a potential \( F \) as follows

\[ F(x, y) = \int_0^x M(s, 0) \, ds + \int_0^y N(x, s) \, ds = \int_0^x 0 \, ds + \int_0^y (-2 \sin x + 3xs^2) \, ds \]

\[ = -2y \sin x + xy^3. \]

Thus the equation is equivalent to

\[ xy^3 - 2y \sin x = C. \]

3. Note that \( \int_0^t e^{2(t-\theta)} x(\theta) \, d\theta \) is the convolution of \( e^{2t} \) and \( x(t) \). Taking Laplace transforms of both sides, we obtain the equation

\[ X(s) = \frac{1}{s^2 + 1} - \frac{1}{s - 2} X(s), \]

where \( X(s) = L\{x\}(s) \) (assuming that \( x \) is continuous and of exponential order). Hence

\[ \left( 1 + \frac{1}{s - 2} \right) X(s) = \frac{1}{s^2 + 1} \]

\[ \Leftrightarrow \frac{s - 2}{s - 2} X(s) = \frac{1}{s^2 + 1} \]

\[ \Leftrightarrow X(s) = \frac{s - 2}{(s^2 + 1)(s - 1)} = \frac{1}{2} \frac{1}{s - 1} + \frac{3}{2} \frac{1}{s^2 + 1} + \frac{1}{2} \frac{s}{s^2 + 1}. \]

Please, turn over!
and \( x(t) = -\frac{1}{2}e^t + \frac{3}{2}\sin t + \frac{1}{2}\cos t \). Since \( x \) is continuous and of exponential order the calculations are justified.

4. We have that \( p_A(\lambda) = \det(A - \lambda I) = -\lambda^2(\lambda + 2) \). The eigenvalues are therefore \( \lambda_1 = -2 \) with algebraic multiplicity 1 and \( \lambda_2 = 0 \) with algebraic multiplicity 2. Solving the equation \( A\bar{x} = -2\bar{x} \), we see that

\[
\bar{v}_{1,1} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

is an eigenvector corresponding to \( \lambda_1 \). Solving the equation \( A\bar{x} = 0 \) we see that

\[
\bar{v}_{2,1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
\]

is an eigenvector corresponding to \( \lambda_2 \). Moreover, \( \lambda_2 \) has geometric multiplicity 1. At this stage, we can compute

\[
A^2 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}
\]

in order to find a basis for \( \mathbb{C}^3 \) consisting of generalised eigenvectors. Such a basis is e.g. given by \( \bar{v}_{1,1}, \bar{v}_{2,1}, \bar{v}_{2,2} \) where

\[
\bar{v}_{2,2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}
\]

However, note that \( \bar{x}_0 = -\bar{v}_{1,1} + \bar{v}_{2,1} \), so it’s actually not necessary to compute this basis. We obtain

\[
\bar{x}(t) = -e^{tA}\bar{v}_{1,1} + e^{tA}\bar{v}_{2,1} = -e^{-2t}\bar{v}_{1,1} + \bar{v}_{2,1} = \begin{pmatrix} 1 \\ -e^{-2t} \\ -e^{-2t} + 1 \end{pmatrix}.
\]

Consequently,

\[
\lim_{t \to \infty} \bar{x}(t) = \bar{v}_{2,1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\]

5. The equation has the form

\[
t^2x'' + tP(t)x' + Q(t)x = 0,
\]

where \( P(t) = 2 \) and \( Q(t) = t^2 \) are analytic at \( t = 0 \) (with \( R = \infty \)). Hence, \( t = 0 \) is a regular singular point. The indicial equation is

\[
F(r) = r(r - 1) + 2r = r(r + 1) = 0,
\]
with roots $r_1 = 0$ and $r_2 = -1$. Setting $x(t) = \sum_{k=0}^{\infty} a_k t^{r+k}$, we obtain
\[
t^2 x(t) = \sum_{k=0}^{\infty} a_k t^{r+k+2} = \sum_{k=2}^{\infty} a_k t^{r+k},
\]
\[
2t x'(t) = \sum_{k=0}^{\infty} 2(r + k) a_k t^{r+k},
\]
\[
t^2 x''(t) = \sum_{k=0}^{\infty} (r + k)(r + k - 1) a_k t^{r+k}.
\]
Comparing coefficients, we get
\[
(r + k)(r + k + 1)a_k = \begin{cases} 0, & k = 0, 1, \\ -a_{k-2}, & k \geq 2. \end{cases}
\]
Taking $r = r_1 = 0$, we get
\[
k(k+1)a_k = \begin{cases} 0, & k = 0, 1, \\ -a_{k-2}, & k \geq 2. \end{cases}
\]
This gives $a_k = 0$ if $k$ is odd, while
\[
2n(2n+1)a_{2n} = -a_{2(n-1)} \iff a_{2n} = -\frac{1}{(2n+1)2n} a_{2(n-1)}.
\]
Thus $a_2 = -\frac{a_0}{3!}$, $a_4 = \frac{a_0}{9!}$ and in general $a_{2n} = (-1)^n \frac{a_0}{(2n+1)!}$ (induction). Taking $a_0 = 1$, we see that
\[
x_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} = \frac{\sin t}{t}
\]
is a Frobenius series solution corresponding to the characteristic exponent 0. Since $r_1 - r_2 = 1$ we can’t be sure that there is a Frobenius series solution corresponding to $r_2$, but we still try to find one. Taking $r = r_2 = -1$, we get
\[
(k-1)ka_k = \begin{cases} 0, & k = 0, 1, \\ -a_{k-2}, & k \geq 2. \end{cases}
\]
For $k = 1$ both sides of the equation are zero, so we can choose $a_1 = 0$. With this choice, we find that $a_k = 0$ whenever $k$ is odd. For $k = 2n$, we get
\[
(2n-1)2na_{2n} = -a_{2(n-1)} \iff a_{2n} = -\frac{a_{2(n-1)}}{(2n-1)2n}.
\]
Using induction we obtain $a_{2n} = (-1)^n \frac{a_0}{(2n)!}$ and thus
\[
x_2(t) = t^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \frac{\cos t}{t}
\]
is a Frobenius series solution corresponding to the characteristic exponent 0. The general solution is thus given by
\[
x(t) = c_1 \frac{\sin t}{t} + c_2 \frac{\cos t}{t}.
\]
Remark: An alternative way of solving the problem is by making the change of variables $x(t) = \frac{y(t)}{t}$. One then obtains the equation $y''(t) + y(t) = 0$, and from this the formula for the solution follows easily.

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6. **a)** We have $u'(t) - \alpha u(t) \leq 0$. Multiplying both sides by $e^{-\alpha t}$, we obtain that

$$\frac{d}{dt}(e^{-\alpha t}u(t)) \leq 0.$$ 

Integrating this inequality gives

$$e^{-\alpha t}u(t) - u(0) \leq 0$$

or

$$u(t) \leq u(0)e^{\alpha t}.$$ 

**b)** Since $f$ is continuous, there exists for each $T > 0$ a constant $\alpha := \max_{0 \leq t \leq T} |f(t)|$ such that $|f(t - \theta)| \leq \alpha$ for $0 \leq t \leq T$. This gives

$$|x(t)| \leq \alpha \int_{0}^{t} |x(\theta)| d\theta := \alpha u(t).$$

We furthermore have that

$$u'(t) = |x(t)| \leq \alpha u(t).$$

By part **a)**, we therefore have

$$u(t) \leq u(0)e^{\alpha t} = 0.$$ 

**c)** Suppose that $x(t)$ and $y(t)$ are both solutions. Then $z(t) := x(t) - y(t)$ satisfies

$$z(t) = \int_{0}^{t} f(t - \theta)x(\theta)\ d\theta,$$

where $f(t) = -e^{2t}$. By part **b)** it follows that $z(t) = 0$ for all $t \geq 0$, so $x(t) \equiv y(t)$. 