Solutions

1. a) We apply the Euclidean algorithm to the numbers 29 and 11 and get

\[
\begin{align*}
29 &= 11 \cdot 2 + 7, \\
11 &= 7 \cdot 1 + 4, \\
7 &= 4 \cdot 2 - 1.
\end{align*}
\]

From this we see that the greatest common divisor of the two numbers is 1. Hence the equation is soluble. We work backwards in the algorithm.

\[
\begin{align*}
7 &= 29 - 11 \cdot 2, \\
4 &= 11 - 7 \cdot 1 = 11 - (29 - 11 \cdot 2) = 29(-1) + 11 \cdot 3, \\
1 &= 4 \cdot 2 - 7 = 2(29(-1) + 11 \cdot 3) - (29 - 11 \cdot 2) = 29(-3) + 11 \cdot 8.
\end{align*}
\]

Multiplying the members of the last equality by 200 yields

\[
200 = 29(-600) + 11 \cdot 1600
\]

and we find that \(x = x_0 = -600\), \(y = y_0 = 1600\) is a particular solution of our equation. According to the theorem on Diophantine equations, the solutions are therefore given by \(x = -600 + 11n, y = 1600 - 29n, n \in \mathbb{Z}\).

**Answer:** \(x = -600 + 11n, y = 1600 - 29n, n \in \mathbb{Z}\).

b) That \(x\) and \(y\) in the above solution are positive means that

\[
n > \frac{600}{11} = 54 + \frac{6}{11} \quad \text{and} \quad n < \frac{1600}{29} = 55 + \frac{5}{29}
\]

and the only integer \(n\) satisfying these conditions is \(n = 55\). The corresponding solution is \(x = 5, y = 5\).

**Answer:** \(x = 5, y = 5\).

2. The modulus of the right-hand side equals

\[
\frac{|1 + i|^5}{|1 + \sqrt{3} i|^2 |1 - i|^2} = \frac{(\sqrt{2})^5}{2^3 (\sqrt{2})^2} = 2^{-3/2}
\]

and its argument is

\[
5 \arg (1 + i) - 3 \arg \left(1 + \sqrt{3} i\right) - 2 \arg (1 - i) = 5 \cdot \frac{\pi}{4} - 3 \cdot \frac{\pi}{3} - 2 \cdot \frac{\pi}{4} = \frac{3\pi}{4}.
\]
We may therefore rewrite the right-hand side of the equation as \(2^{-3/2}e^{3\pi i/4}\), and if we set \(z = re^{i\theta}\), we get the equivalent equation
\[
r^3e^{3\theta} = 2^{-3/2}e^{3\pi i/4}.
\]
From this it follows that \(r = 2^{-1/2}\) and \(\theta = \pi/4 + 2\pi k/3\), where \(k\) is an integer. Hence the solutions are \(z = 2^{-1/2}e^{(\pi/4+2\pi k/3)i}\), \(k = 0, 1, 2\).

**Answer:** \(z = z_k = 2^{-1/2}e^{(\pi/4+2\pi k/3)i}, k = 0, 1, 2\).

3. **a)** Using the binomial theorem we have that
\[
\left(x^2 - \frac{1}{2x}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} \left(x^2\right)^{9-k} \left(-\frac{1}{2x}\right)^k = \sum_{k=0}^{9} \binom{9}{k} (-1)^k 2^{-k} x^{18-3k}.
\]
The term with index \(k\) is constant if and only if \(18 - 3k = 0\), i.e. \(k = 6\). Hence the constant term equals
\[
\binom{9}{6} (-1)^6 2^{-6} = \binom{9}{3} 2^{-6} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 2^6} = \frac{21}{16}.
\]
**Answer:** \(21/16\).

**b)** There are 11 letters in the word TALLAHASSEE. If we consider the letters to be distinguishable, there are \(11!\) arrangements. There are 3 A’s, 2 E’s, 2 L’s, 2 S’s, 1 H, and 1 T in the word. Hence each distinct arrangement has been counted \(3!2!2!2!\) times and so the number of distinct arrangements is

\[
\frac{11!}{3!2!2!2!} = 831600.
\]
**Answer:** 831600.

4. If \(z\) is a real number, then the real and imaginary parts of the left-hand side of the equation are
\[
z^3 - z^2 - 6z + 8 \quad \text{and} \quad -3z^2 + 3z + 6,
\]
respectively. The zeros of the latter expression are \(z = -1\) and \(z = 2\). Checking reveals that only \(z = 2\) is a zero of the first expression. Hence \(z = 2\) is the only real root of the equation. The quotient after division of \(z^3 - (1 + 3i)z^2 - (6 - 3i)z + 8 + 6i\) by \(z - 2\) is \(z^2 + (1 - 3i)z - 4 - 3i\). It remains to solve the equation
\[
z^2 + (1 - 3i)z = 4 + 3i.
\]
Completing the square gives the equivalent equation
\[
\left(z + \frac{1-3i}{2}\right)^2 = 4 + 3i + \left(\frac{1-3i}{2}\right)^2 = 4 + 3i + \frac{-8-6i}{4} = 2 + \frac{3i}{2}.
\]
Setting \(x + iy = z + (1-3i)/2\), identifying the real and imaginary parts and equating the moduli of both sides of the equation give
\[
\begin{align*}
x^2 - y^2 &= 2, \\
2xy &= \frac{3}{2}, \\
x^2 + y^2 &= \frac{5}{2}.
\end{align*}
\]
Equations (1) and (3) yield \(x^2 = 9/4\) and \(y^2 = 1/4\). Hence the possible solutions are given by \(x + iy = \pm(3/2 + i/2)\). By (2), \(x\) and \(y\) have the same sign. Therefore only \(x + iy = \pm(3/2 + i/2)\) qualify as solutions. Thus the roots of the quadratic equation are given by
\[
z = \frac{3+i - 1-3i}{2} = 1+2i \quad \text{or} \quad z = \frac{-3+i-1-3i}{2} = -2 + i.
\]
**Answer:** The roots are 2, 1 + 2\(i\) and \(-2 + i\).

5. When \(n = 1\), both sides of the inequality are equal to 1, whence the inequality holds in this case. Suppose that the inequality holds when \(n = p\) is a positive integer. We shall then show that it holds also when \(n = p + 1\). Using the induction hypothesis we get that
\[
\begin{align*}
\sum_{k=1}^{p+1} k^2 &= \sum_{k=1}^{p} k^2 + (p+1)^2 \\
&\geq \frac{p(p+1)^2}{4} + (p+1)^2 = \frac{(p+4)(p+1)^2}{4} \\
&= (p+4)(p+1) \cdot \frac{p+1}{4}.
\end{align*}
\]
When \(n = p + 1\), the right-hand side of the inequality equals
\[
\frac{(p+1)(p+2)^2}{4} = (p+2)^2 \cdot \frac{p+1}{4}.
\]
Hence it is sufficient to show that
\[
(p+4)(p+1) \geq (p+2)^2, \quad p \geq 1,
\]
and this follows from the fact that
\[
(p+4)(p+1) - (p+2)^2 = p^2 + 5p + 4 - p^2 - 4p - 4 = p \geq 0.
\]
This concludes the proof by induction.

6. a) Since \(b\) is odd, we have that \(a^b = -(-a)^b\) and therefore that
\[
a^b + 1 = -((-a)^b - 1) = -(-a - 1) \left(1 + (-a) + (-a)^2 + \cdots + (-a)^{b-1}\right) \\
= (a + 1) \left(1 + (-a) + (-a)^2 + \cdots + (-a)^{b-1}\right).
\]
Hence \(a + 1\) divides \(a^b + 1\).
b) Assume that $k$ is not a power of 2. Then $k = 2^n \cdot b$ for some natural number $n$ and some odd integer $b \geq 3$. Therefore $2^k + 1 = 2^{2^n \cdot b} + 1 = (2^{2^n})^b + 1$ is divisible by $2^{2^n} + 1$. Since $b \geq 3$, we have that $1 < 2^{2^n} + 1 < (2^{2^n})^b + 1 = 2^k + 1$. This shows that $2^{2^n} + 1$ is a non-trivial divisor of $2^k + 1$, which therefore is not a prime number.