Problem 1. The solution of the recurrence problem is:
\[ a_n = (1 - i)(1 + i)^n + (1 + i)(1 - i)^n + n, \quad n = 0, 1, 2, \ldots \]
□

Problem 2. The solutions of the congruences are:
\[ x = 59 + 154n, \quad n \in \mathbb{Z}. \]
□

Problem 3. We know that there is a one-to-one correspondence between equivalence relations on a set \( X \) and partitions of the same set \( X \). The number of partitions of the set \( \{1, 2, 3, 4\} \) is
\[ \sum_{k=1}^{4} S(4, k) = 1 + 7 + 6 + 1 = 15, \]
where \( S(n, k) \) denotes Stirling numbers of the second kind.
□

Problem 4. We shall use the so-called principle of inclusion and exclusion. Denote by \( X \) the set of all integers \( n \) such that \( 1 \leq n \leq 123 \). Denote by \( X_2 \) the subset of \( X \) consisting of the even integers in \( X \), denote by \( X_3 \) the subset of integers in \( X \) divisible by 3, and denote by \( X_7 \) the set of integers in \( X \) that are divisible by 7. Phrased in these terms we want to calculate the number of integers in the set \( X \setminus (X_2 \cup X_3 \cup X_7) \).

By the above mentioned principle we have that
\[
|X \setminus (X_2 \cup X_3 \cup X_7)| = |X| - (|X_2| + |X_3| + |X_7|) \\
+ (|X_2 \cap X_3| + |X_2 \cap X_7| + |X_3 \cap X_7|) - |X_2 \cap X_3 \cap X_7|,
\]
where the symbol \( |\cdot| \) is used to indicate the number of elements in a set. A calculation gives that \( |X_2| = 61, |X_3| = 41 \) and \( |X_7| = 17 \). Next observe that the set \( X_2 \cap X_3 \) consists of all integers in \( X \) divisible by 6. Arguing this way we see that \( |X_2 \cap X_3| = 20, |X_2 \cap X_7| = 8, |X_3 \cap X_7| = 5, \) and \( |X_2 \cap X_3 \cap X_7| = 2 \). We now have that the number of integers \( 1 \leq n \leq 123 \) not divisible by 2, 3 or 7 equals
\[ |X \setminus (X_2 \cup X_3 \cup X_7)| = 123 - (61 + 41 + 17) + (20 + 8 + 5) - 2 = 35. \]
□

Problem 5. Notice first the prime factorization \( 20000 = 2^5 5^4 \), and that every positive integer \( n_j \) must have a prime factorization of the form \( n_j = 2^{k_j} 5^{l_j} \) for some nonnegative integers \( k_j \) and \( l_j \) (\( j = 1, 2, 3 \)). By the fundamental theorem of arithmetic we have that the triple \( (n_1, n_2, n_3) \) is such that \( n_1 n_2 n_3 = 20000 \) if and only if the triple \( (k_1, k_2, k_3) \) satisfies
\[
\begin{align*}
\begin{cases}
  k_1 + k_2 + k_3 = 5, \\
  k_1 \geq 0, \quad k_2 \geq 0, \quad k_3 \geq 0,
\end{cases}
\end{align*}
\]
(1)
and the triple \((l_1, l_2, l_3)\) satisfies
\[
\begin{align*}
& l_1 + l_2 + l_3 = 4, \\
& l_1 \geq 0, \ l_2 \geq 0, \ l_3 \geq 0.
\end{align*}
\]
By standard theory (Section I.1.4 in Grimaldi) problem (1) has \(\binom{5+3-1}{5} = \binom{7}{5}\) solutions \((k_1, k_2, k_3)\) and problem (2) has \(\binom{6}{4}\) solutions \((l_1, l_2, l_3)\). As a result our problem has \(\binom{7}{5}\binom{6}{4} = 315\) solutions \((n_1, n_2, n_3)\).

\[\square\]

**Problem 6.** Passing from the recurrence formula for the Fibonacci numbers to the exponential generating function we see that \(F\) solves the second order constant coefficient differential equation initial value problem
\[
\begin{cases}
F'' = F' + F, \\
F(0) = 1, \quad F'(0) = 1,
\end{cases}
\]
where the prime \('\) indicates derivative. A calculation gives that
\[
F(x) = \frac{1 + \sqrt{5}}{2\sqrt{5}} \exp\left(\frac{1 + \sqrt{5}}{2} x\right) - \frac{1 - \sqrt{5}}{2\sqrt{5}} \exp\left(\frac{1 - \sqrt{5}}{2} x\right),
\]
where \(\exp(x) = e^x\) is the usual exponential function.

\[\square\]