1. The characteristic polynomial is \( r^2 + r - 6 = (r + 3)(r - 2) \). Therefore, the solutions to the homogeneous, and particular equations are of the form
\[
a_{n}^{(h)} = \alpha (-3)^n + \beta 2^n \\
a_{n}^{(p)} = \delta 3^n + \gamma 2^n n.
\]
After some computations, we should arrive at the solution
\[
a_n = 3(-3)^n - 2^{n+1} + 3^{n+1} + 2^n n = 3^{n+1}(1 + (-1)^n) + 2^n(n - 2).
\]

2. Compute \( xG \) for \( x \in \mathbb{Z}_3^3 \) to obtain the list of all code-words and their weights:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( xG )</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>000</td>
<td>0000000</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>1000111</td>
<td>4</td>
</tr>
<tr>
<td>010</td>
<td>0101101</td>
<td>4</td>
</tr>
<tr>
<td>001</td>
<td>0011011</td>
<td>4</td>
</tr>
<tr>
<td>110</td>
<td>1101010</td>
<td>4</td>
</tr>
<tr>
<td>101</td>
<td>1011100</td>
<td>4</td>
</tr>
<tr>
<td>011</td>
<td>0110110</td>
<td>4</td>
</tr>
<tr>
<td>111</td>
<td>1110001</td>
<td>4</td>
</tr>
</tbody>
</table>

Since the smallest weight is equal to the Hamming separation \( d(C) \) of the code, we get \( d(C) = 4 \). Also, it is clear that 1101010 is in the code, and 1010100 is not. The code-word 1011100 is the unique code-word at a Hamming distance of 1 from 1010100, and is therefore its correction.

3. Following the scheme of the Chinese Remainder theorem, we let \( N_1 = 28, N_2 = 21, N_3 = 12 \), and solve
\[
s_1 \cdot 28 \equiv 1 \pmod{3} \quad s_1 \equiv 1 \pmod{3} \\
s_1 \cdot 21 \equiv 1 \pmod{4} \quad \Rightarrow \quad s_2 \equiv 1 \pmod{4} \\
s_1 \cdot 12 \equiv 1 \pmod{7} \quad s_3 \equiv 3 \pmod{7}
\]
The smallest positive \( x \) satisfying this is then
\[
x = 1 \cdot 28 \cdot 2 + 1 \cdot 21 \cdot 3 + 3 \cdot 12 \cdot 3 = 227 \equiv 59 \pmod{84}.
\]

4. We use the principle of inclusion and exclusion. Let \( N = 6! \) denote the number of ways to arrange 1, 2, 3, 4, 5, 6 along a line, and consider the conditions
\[
c_1 = 12 \text{ appears,} \\
c_2 = 23 \text{ appears,} \\
c_3 = 34 \text{ appears.}
\]
We calculate
\[
N(c_1) = N(c_2) = N(c_3) = 5! \\
N(c_1 c_2) = N(c_2 c_3) = N(c_1 c_3) = 4! \\
N(c_1 c_2 c_3) = 3!.
\]
In the calculation of \( N(c_1) \), say, we treat ‘12’ as being one object. In the calculation of \( N(c_1 c_2) \) we have to treat ‘123’ as one object, while in the calculation of \( N(c_1 c_3) \) we treat each of ‘12’ and ‘34’ as one object.
So, by the principle of inclusion and exclusion, our answer is
\[
N(c_1 c_2 c_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1 c_2) + N(c_1 c_3) + N(c_2 c_3)] - N(c_1 c_2 c_3)
\]
= 6! − 3 · 5! + 3 · 4! − 3! = 426.

5.

a. We check that \( p(x) = x^4 + 2x^3 + x^2 + 1 \) is a prime polynomial. First, we exclude the possibility of it having a linear factor by observing that it has no zeroes: \( p(0) = 1, \ p(1) = 2, \ p(2) = 1 \) in \( \mathbb{Z}_3 \). The only possibility for it to be reducible is if it is a product of two irreducible polynomials of degree 2.

We consider all polynomials of the form \( f(x) = x^2 + \alpha x + \beta \). (Note that a polynomial \( 2x^2 + \alpha x + \beta \) can be made to be of this form by multiplying with \( 2^{-1} = 2 \).) Such a polynomial is irreducible if and only if it has no zeroes. This implies that \( \beta \neq 0 \). We make a table

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( f(x) )</th>
<th>zeroes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( x^2 + 1 )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>( x^2 + 2 )</td>
<td>( x = 1, x = 2 )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( x^2 + x + 1 )</td>
<td>( x = 1 )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( x^2 + x + 2 )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( x^2 + 2x + 1 )</td>
<td>( x = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( x^2 + 2x + 2 )</td>
<td></td>
</tr>
</tbody>
</table>

We now compute all products of the irreducible polynomials of degree 2:

\[
\begin{align*}
(x^2 + 1)^2 &= x^4 + 2x^2 + 1 \\
(x^2 + x + 2)^2 &= x^4 + 2x^3 + 2x^2 + x + 1 \\
(x^2 + 2x + 2)^2 &= x^4 + x^3 + 2x + 1 \\
(x^2 + 1)(x^2 + x + 2) &= x^4 + x^3 + x + 2 \\
(x^2 + 1)(x^2 + 2x + 2) &= x^4 + 2x^3 + 2x + 2 \\
(x^2 + x + 2)(x^2 + 2x + 2) &= x^4 + 1.
\end{align*}
\]

As our \( p(x) \) is not in this list, and has no zeroes, it has to be irreducible. Hence, \( \mathbb{Z}_3[x]/p(x) \) is a field.

b. By the division algorithm, if \( f(x) \) is any polynomial, there exists polynomials \( q(x) \) and \( r(x) \) such that \( f(x) = q(x)p(x) + r(x) \), and \( r(x) \) has strictly lower degree than \( p(x) \). This answers the first question.

To compute the equivalence class of \( x^5 + 1 \), we use long division to find that \( x^5 + 1 = (x + 1)(x^4 + 2x^3 + x^2 + 1) + (2x^2 + 2x) \). So \( [x^5 + 1] = [2x^2 + 2x] \) in \( \mathbb{Z}_3[x] \).

Finally, to compute the inverse of \( x^5 + 1 \), it is enough to compute the inverse of \( 2x^2 + 2 \). This can be done by using the Euclidean algorithm, and we find (by just one step)

\[
(x^4 + 2x^3 + x^2 + 1) = (2x^2 + 2x)(2x^2 + 2x) + 1.
\]

Hence \( [2x^2 + 2x]^{-1} = -[2x^2 + 2x] = [2x^2 + x] \).

6.

a. For this problem, we need to find the coefficient of \( x^6/6! \) of the exponential generating function

\[
(1 + x + x^2/2)(1 + x)^5.
\]

By the binomial theorem, this is the same as

\[
\sum_{k=0}^{5} \binom{5}{k} x^k (1 + x + x^2/2),
\]

and so the answer is

\[
6! \left[ \binom{5}{5} + \binom{5}{1} \frac{1}{2} \right] = 7! / 2 = 2520.
\]
b. We first observe that with the letters 'NUNBET' we can construct $6!/2$ combinations of length six. To find the remaining combinations, where we all the time use the blank tile, consider 3 cases.

Case 1: The blank tile is none of N, U, B, E, T.

Case 2: The blank tile is one of U, B, E, T.

Case 3: The blank tile is N.

Observe that in all cases we need to assume that the blank tile is used to avoid overcounting!

Case 1: There are 21 such choices for the blank tile. For each choice, the number of combinations where we use the blank tile, is equal to the coefficient of $x^{6}/6!$ in $x(1 + x)^{4}(1 + x + x^{2}/2)$, which is

$$\left[ \binom{4}{4} + \binom{4}{3} \right] 6! = 3 \cdot 6!.$$  

Case 2: There are 4 such choices. For each, the number of combinations is equal to the coefficient of $x^{6}/6!$ in $(x^{2}/2)(1 + x)^{3}(1 + x + x^{2}/2)$, which is

$$\left[ \frac{1}{2} \binom{3}{3} + \frac{1}{4} \binom{3}{2} \right] 6! = (5/4) \cdot 6!.$$  

Case 3: There is only one such choice, and the number of combinations is equal to the coefficient of $x^{6}/6!$ in $(x^{3}/3!)(1 + x)^{4}$, which is:

$$\left[ \frac{1}{3!} \binom{4}{3} \right] 6! = (2/3) \cdot 6!.$$  

The final answer is now (by the rule of sum):

$$6! \cdot \left[ \frac{1}{2} + 21 \cdot 3 + 4 \cdot \frac{5}{4} + 1 \cdot \frac{2}{3} \right] = 6! \cdot \frac{415}{6} = 49.800.$$