1. Let
\[ u(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & 1 \leq x < \pi.
\end{cases} \]

a) Find the cosine series of \( u \).
b) Find the sum of the series
\[ \sum_{k=1}^{\infty} \frac{\sin(2k)}{k}. \]

2. Find a solution \( u \) of the following problem
\[
\begin{align*}
\partial_t u(x, t) &= \partial_x^2 u(x, t) & \text{when } t > 0 \text{ and } 0 < x < \pi, \\
\partial_x u(0, t) &= \partial_x u(\pi, t) = 0 & \text{when } t > 0, \\
u(x, 0) &= \sin^2 x & \text{when } 0 < x < \pi.
\end{align*}
\]

3. Find a function \( u \) such that
\[
\int_{-\infty}^{\infty} \frac{u(x-y)}{1+y^2} dy = \frac{1}{4+x^2}.
\]

4. Let \( f(x) = xe^{-x} \) for \( x > 0 \) and \( f(x) = 0 \) for \( x \leq 0 \).
a) Find the Fourier transform of \( f \).
b) For each \( \lambda > 0 \), find the value of the integral
\[
\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x(1+ix)^2} dx.
\]

5. Let \( u(x) = x \) when \( 0 \leq x \leq \pi \).
a) Determine the numbers \( \lambda_n \) so that, for each positive integer \( N \), the integral
\[
\int_{0}^{\pi} |u(x) - \sum_{n=1}^{N} \lambda_n \sin(2nx)|^2 dx
\]
is as small as possible.
b) For these values of \( \lambda_n \), find the limit of the integral in a) as \( N \to \infty \).

\textbf{Hint}: One might want to consider the cosine series of the function \( v(x) = x^2 \), \( 0 \leq x \leq \pi \).