1. Let $g: \mathbb{N} \to [0, +\infty]$, where $\mathbb{N}$ denotes the set of positive integers, and define the set function $\mu_g$ by

$$\mu_g(A) = \sum_{n \in A} g(n), \quad A \subseteq \mathbb{N}$$

(recall that an empty sum is defined to be 0).

a) Prove that $\mu_g$ defines a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, where $\mathcal{P}(\mathbb{N})$ is the power set of $\mathbb{N}$.

b) For which $g$ is $\mu_g$ $\sigma$-finite?

c) For which $g$ is $\mu_g$ finite?

d) Let $f: \mathbb{N} \to [0, +\infty)$. Show that $f$ is measurable and that

$$\int f \, d\mu_g = \sum_{n=1}^{\infty} f(n)g(n).$$

2. Compute the following limits, if they exist.

a) $\lim_{n \to \infty} \int_0^{\infty} (\cos x)^n e^{-x} \, dx$,

b) $\lim_{n \to \infty} \int_0^{\infty} e^{-\frac{n}{1+x}} \, dx$.

3. a) Show that $L^4([0,1]) \subseteq L^2([0,1])$.

b) Show by counterexample that $L^4(\mathbb{R}) \not\subseteq L^2(\mathbb{R})$.

c) Show that $L^4(\mathbb{R}) \cap L^1(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

Hint: in problem c) it might be useful to consider the sets $\{ x \in \mathbb{R} : |f(x)| \leq 1 \}$ and $\{ x \in \mathbb{R} : |f(x)| > 1 \}$ separately.
4. Use Fubini’s theorem and the relation

\[ \frac{1}{x} = \int_0^\infty e^{-xt} \, dt, \quad x > 0, \]

to prove that

\[ \lim_{A \to \infty} \int_0^A \frac{\sin x}{x} \, dx = \frac{\pi}{2}. \]

5. a) Let \((X, \mathcal{A}, \mu)\) be a measure space and let \(\{A_n\}\) be a sequence of sets belonging to \(\mathcal{A}\). Prove that if \(\sum_{n=1}^{\infty} \mu(A_n) < \infty\), then \(\mu(\lim_n A_n) = 0\), where \(\lim_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\). This is known as the Borel-Cantelli lemma.

b) Let \(f: X \to \mathbb{R}\) and \(f_n: X \to \mathbb{R}, \quad n = 1, 2, \ldots\), be \(\mathcal{A}\)-measurable functions and assume that \(\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) < \infty\) for any \(\varepsilon > 0\). Show that \(f_n \to f\) \(\mu\)-a.e.

c) Give a counterexample showing that the result in part b) is not true if the assumption is relaxed to \(\lim_n \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0\) for any \(\varepsilon > 0\).