1. Let $\mu$ be the counting measure on $\mathbb{N}$, that is, $\mu(E)$ equals the number of elements of $E$ if $E$ is finite and $\mu(E) = \infty$ otherwise. Suppose that $(f_n)$ is a sequence of real-valued (obviously $\mu$-measurable) functions on $\mathbb{N}$ that converges in measure to $f$. Show that $(f_n)$ converges uniformly to $f$.

2. a) Let $X$ be an infinite set and let $\mu$ be a finite outer measure on the subsets of $X$ such that every set $\{x\}$, $x \in X$ is $\mu$-measurable and there exists a countable subset $A_\mu$ of $X$ with $\mu(X - A) = 0$. Show that $\mu$ is a measure on the $\sigma$-algebra of all subsets of $X$.

b) Let $\mu, \nu$ be finite outer measures on $X$ with the property in a), that is, there exist countable sets $A_\mu, A_\nu \subset X$ with $\mu(X - A_\mu) = \nu(X - A_\nu) = 0$. Find the Lebesgue decomposition of $\nu$ with respect to $\mu$.

3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $1 < p < \infty$. Suppose that $(f_n)$ is a bounded sequence in $L^p(X, \mu)$ (that is, $\sup_n \|f_n\|_p < \infty$), such that $(f_n)$ converges in measure to $f$.

a) Prove that $f \in L^p(x, \mu)$.

b) Show that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$.

4. a) Find an unbounded Lebesgue measurable function $f$ on $[0,1]$ with

$$\int_{[0,1]} |f(x)|^p dx < \infty$$

for all $p > 1$.

5. Show that the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(x\sqrt{n})}{n^2}$$
is differentiable on $\mathbb{R}$ and compute its derivative.

6. Let $f, g : \mathbb{R}^n \mapsto \mathbb{R}$ be integrable functions with respect to the Lebesgue measure $m_n$ on $\mathbb{R}^n$. Show that for $m_n$-almost every $y \in \mathbb{R}^n$ the function $x \mapsto f(x + y)g(x)$ is integrable. Prove also that

$$y \mapsto \int_{\mathbb{R}^n} f(x + y)g(x)dm_n(x)$$

is $m_n$-integrable.