Solutions

1. First we determine the kernel:

\[
\begin{align*}
-x + y + z - w &= 0 \\
2x + 3y - z &= 0 \\
4x + y + z - 2w &= 0
\end{align*}
\]

\[\Leftrightarrow \begin{align*}
x - y + z - w &= 0 \\
5y - 3z + 2w &= 0
\end{align*}\]

The solution is \((x, y, z, w) = (-2s + 3t, 3s - 2t, 5s, 5t), s, t \in \mathbb{R}\) Thus the kernel is of dimension 2 and the vectors \((-2, 3, 5, 0)\) and \((3, -2, 0, 5)\) are a basis for the kernel.

By the dimension theorem, we have

\[
\text{dim}(\ker(F)) + \text{dim}(\text{range}(F)) = 4,
\]

so the dimension of the range is 2. For example the first two columns of the matrix, the vectors \((1, 2, 4)\) and \((-1, 3, 1)\), belong to the range and they are linearly independent since they are not parallel. Then they are a basis for the range because any set of \(n\) linearly independent vectors in an \(n\)-dimensional space is a basis.

**Answer:** One correct answer is: The vectors \((-2, 3, 5, 0)\) and \((3, -2, 0, 5)\) are a basis for the kernel. The vectors \((1, 2, 4)\) and \((-1, 3, 1)\) are a basis for the range. There are infinitely many other correct answers.

2. The points are on the line \(y = at + b\) if

\[
\begin{align*}
-a + b &= 2 \\
b &= 2 \\
a + b &= 1 \\
2a + b &= 0
\end{align*} \implies A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} = Y
\]

To obtain the best least squares fit we multiply the equation by \(A^T\) and get

\[
A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T Y \iff \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.
\]

The solution is \(a = -0.7, b = 1.6\).

**Answer:** The line with the best least squares fit to the points is \(y = -0.7t + 1.6\).

3. The vectors \(\mathbf{v}_1\) and \(\mathbf{v}_2\) are linearly independent and they are a basis for the plane \(V = \text{span}[\mathbf{v}_1, \mathbf{v}_2]\). We orthogonalize \(\mathbf{v}_1\) and \(\mathbf{v}_2\) to obtain an orthonormal basis for \(V\):

\[
\begin{align*}
\mathbf{e}_1 &= \mathbf{v}_1 / \|\mathbf{v}_1\| = \frac{1}{\sqrt{2}} (1, 1, 0) \\
\mathbf{f}_2 &= \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{e}_1) \mathbf{e}_1 = (3, 1, 1) - \frac{4}{2} (1, 1, 0) = (1, -1, 1) \\
\mathbf{e}_2 &= \mathbf{f}_2 / \|\mathbf{f}_2\| = \frac{1}{\sqrt{3}} (1, -1, 1)
\end{align*}
\]

*Please, turn over!*
To extend \( e_1 \) and \( e_2 \) to an orthonormal basis for \( \mathbb{R}^3 \), we can take any vector that is orthogonal to \( v_1 \) and \( v_2 \), for example \((1, -1, 2)\), and divide by the length. Thus we choose \( e_3 = (1, -1, -2)/\sqrt{6} \).

**Answer:** The vectors \( e_1 = \frac{1}{\sqrt{2}}(1, 1, 0) \) and \( e_2 = \frac{1}{\sqrt{3}}(1, -1, 1) \) are an orthonormal basis for \( V \). With \( e_3 = (1, -1, -2)/\sqrt{6} \), the vectors \( e_1, e_2 \) and \( e_3 \) are a basis for \( \mathbb{R}^3 \).

4. A matrix is a rotation if and only if it is orthogonal and the determinant is 1. The columns of the matrix \( A \) are orthogonal vectors with length 7 and the determinant is positive. Therefore we get an rotation matrix if we multiply \( A \) by \( 1/7 \). To find the rotation axis we solve the system \( cAX = X \iff AX = 7X \) which is equivalent to

\[
\begin{align*}
-13x + 2y + 3z &= 0 \\
2x - 10y + 6z &= 0 \\
3x + 6y - 5z &= 0
\end{align*}
\]

which has the solution \( t(1, 2, 3) \). To find the rotation angle, we choose any vector that is orthogonal to the rotation axis, for example \( v = (2, -1, 0) \). Then \( \frac{1}{7}Av = -v \) so the angle is \( \pi \).

**Answer:** The matrix \( cA \) is a rotation if and only if \( c = 1/7 \). The rotation axis is \( t(1, 2, 3) \) and the rotation angle is \( \pi \).

5. The quadratic form is given by the symmetric matrix

\[
A = \begin{pmatrix}
0 & 2 & 2 \\
2 & 3 & -1 \\
2 & -1 & 3
\end{pmatrix}.
\]

The eigenvalues of \( A \) are the solutions of the characteristic equation

\[
\begin{vmatrix}
-\lambda & 2 & 2 \\
2 & 3-\lambda & -1 \\
2 & -1 & 3-\lambda
\end{vmatrix} = \begin{vmatrix}
4-\lambda & 4-\lambda & 4-\lambda \\
2 & 3-\lambda & -1 \\
2 & -1 & 3-\lambda
\end{vmatrix} = (4-\lambda)(4-\lambda)(-2-\lambda) = 0.
\]

Thus the eigenvalues are \( \lambda = 4 \) (double) and \( \lambda = -2 \). This implies that we have

\[
Q(x, y, z) = 4\hat{x}^2 + 4\hat{y}^2 - 2\hat{z}^2,
\]

where \( (\hat{x}, \hat{y}, \hat{z}) \) are the coordinates of the vector \((x, y, z)\) with respect to an ON-basis so

\[
\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = x^2 + y^2 + z^2.
\]

It follows that

\[
-2(x^2 + y^2 + z^2) = -2(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) \leq Q(x, y, z) \leq 4(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) = 4(x^2 + y^2 + z^2)
\]

So the maximal value of \( Q \) when \( x^2 + y^2 + z^2 = 1 \) is 4 and the minimal value is \(-2\).

**Answer:** Maximal value 4 and minimal value \(-2\).

6. The rank of the matrix is 4 for all \( x \) such that \( \det(A) \neq 0 \). We subtract the first row from all the following rows and get

\[
\det(A) = \begin{vmatrix}
x & 1 & 1 & 1 \\
1-x & x-1 & 0 & 0 \\
1-x & 0 & x-1 & 0 \\
1-x & 0 & 0 & x-1
\end{vmatrix}.
\]
Now we can factor out \((x - 1)\) from all the rows except the first and after that add the columns number two to four to the first one. This gives:

\[
det(A) = (x - 1)^3
\begin{vmatrix}
  x & 1 & 1 & 1 \\
  -1 & 1 & 0 & 0 \\
  -1 & 0 & 1 & 0 \\
  -1 & 0 & 0 & 1 \\
\end{vmatrix} = (x - 1)^3(\begin{vmatrix}
  x + 3 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{vmatrix}) = (x - 1)^3(x + 3).
\]

Thus the rank is 4 for all \(x\) except 1 and \(-3\). When \(x = 1\), all the columns of \(A\) are equal so the rank is 1. The rank of a matrix is not changed by column operations or row operations. So to find the rank of \(A\) when \(x = -3\) we can check the last matrix in the calculation above with \(x = -3\), namely

\[
\begin{pmatrix}
  0 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

which clearly has 3 independent rows (and columns) and therefore rank equal to 3. **Answer:** The rank is 1 for \(x = 1\), 3 for \(x = -3\) and 4 for all other values.