Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

No books, notes, computational devices etc. are allowed. Use paper supplied by the Department, write only on one side of each paper, and treat at most one exercise on each paper. Use clear handwriting and give clear careful motivations. Fill in the form completely and write your name on each sheet of paper.

1. Find a function \( f \in L^2(\mathbb{T}) \) of minimal norm

\[
\|f\|_2 = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta \right)^{1/2}
\]

such that \( \hat{f}(n) = 1/n \) for \( n = 1, 2, 3, \ldots \), or show that no such minimizer exists. Here

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots,
\]

are the Fourier coefficients of \( f \).

2. Let \( X \) be an infinite-dimensional normed space. Let \( \ell_1, \ldots, \ell_n \) be continuous linear functionals on \( X \) and consider the set

\[
U = \left\{ x \in X : |\ell_j(x)| < 1 \text{ for } 1 \leq j \leq n \right\}.
\]

Show that the set \( U \) is unbounded.

3. Let \( X \) be a Banach space. Let \( X_1 \) and \( X_2 \) be two closed subspaces of \( X \) such that every element \( x \in X \) has a representation

\[
x = x_1 + x_2, \tag{1}
\]

where \( x_1 \in X_1 \) and \( x_2 \in X_2 \). Show that there exists a finite constant \( c \) such that every element \( x \in X \) has a representation (1) with \( x_1 \in X_1, x_2 \in X_2 \) and

\[
\|x_1\| + \|x_2\| \leq c\|x\|.
\]

4. Let \( A \) be a compact positive operator on a Hilbert space \( \mathcal{H} \). Show that the positive square root \( B = A^{1/2} \) of \( A \) is compact.

Var god vänd!
5. Denote by $c_0$ the space of all bounded sequences $\{a_n\}_{n=0}^\infty$ of complex numbers such that $\lim_{n \to \infty} a_n = 0$ normed by
\[
\|\{a_n\}_{n=0}^\infty\|_\infty = \sup_{n \geq 0} |a_n|.
\]
Show that the dual of $c_0$ is naturally identified with the space $\ell^1$. Show also that the space $c_0$ is not reflexive.

6. Let $1 < p < \infty$ and consider the continuous linear functional $\Lambda_n$ on $L^p(0, \infty)$ defined by
\[
\Lambda_n(f) = \int_0^{n\pi} \sin(nx)f(x)\,dx, \quad f \in L^p(0, \infty),
\]
for $n = 1, 2, 3, \ldots$. Determine whether the sequence $\{\Lambda_n\}_{n=1}^\infty$ of continuous linear functionals has a weak* limit in the dual of $L^p(0, \infty)$. Calculate also the weak* limit $\lim_{n \to \infty} \Lambda_n$ if it exists.