Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initier på varje papper.

No books, notes, computational devices etc. are allowed. Use paper supplied by the Department, write only on one side of each paper, and treat at most one excercise on each paper. Use clear handwriting and give clear careful motivations. Fill in the form completely and write your name on each sheet of paper.

1. Denote by $\ell^\infty$ the space of all bounded sequences $\{a_n\}_{n=0}^\infty$ of complex numbers normed by

$$\|\{a_n\}_{n=0}^\infty\|_\infty = \sup_{n \geq 0}|a_n|.$$ 

Show that there exists a bounded linear functional $\Lambda$ on $\ell^\infty$ such that

$$\Lambda(\{a_n\}_{n=0}^\infty) = \lim_{n \to \infty} a_{2n}$$

whenever the sequence $\{a_n\}_{n=0}^\infty \in \ell^\infty$ is such that the limit $\lim_{n \to \infty} a_{2n}$ exists.

2. Determine whether there exists a function $f \in L^2(-1,1)$ such that

$$\int_{-1}^{1} f(t)dt = 0 \quad \text{and} \quad \int_{-1}^{1} tf(t)dt = 1$$

minimizing the quantity

$$\int_{-1}^{1} |f(t) - \log|t||^2dt.$$ 

Determine also whether such a minimizer is uniquely determined (as element in $L^2(-1,1)$) by these conditions if it exists.

3. Consider the function $f_n$ on the positive half axis $(0, \infty)$ defined by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } 0 < x < 1/n, \text{ and} \\ 0 & \text{if } 1/n \leq x < \infty, \end{cases}$$

for $n = 1, 2, 3, \ldots$. Determine whether the sequence $\{f_n\}_{n=1}^\infty$ is weakly convergent in $L^2(0,\infty)$. Determine also whether the sequence $\{f_n\}_{n=1}^\infty$ is norm convergent in $L^2(0,\infty)$.

Var god vänd!
4. Let $X$ be a Banach space. Let $X_1$ and $X_2$ be two closed subspaces of $X$ such that $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$. Consider the map $P_1$ on $X$ defined by

$$P_1 x = x_1$$

if $x = x_1 + x_2$, where $x_1 \in X_1$ and $x_2 \in X_2$. Show that $P_1$ so defined is a continuous linear map on $X$. Here $X_1 + X_2$ is the vector sum

$$X_1 + X_2 = \{x_1 + x_2 : x_1 \in X_1 \text{ and } x_2 \in X_2\}.$$

5. Consider the Volterra operator

$$Vf(x) = \int_0^x f(t)dt, \quad x \in (0, 1),$$

acting as a bounded operator on $L^2(0, 1)$ into itself. Show that the operator $V + V^*$ is an orthogonal projection and determine the subspace onto which it projects.

6. Let $T$ be a compact normal operator on a Hilbert space $\mathcal{H}$, and let $f$ be a continuous function on the spectrum $\sigma(T)$ for $T$. Show that the operator $f(T)$ is compact if and only if $f(0) = 0$. 