Please write on the front page your preferred time for oral exam.

1. Let $X$ be a Banach space.
   (a) If $(x_n)_{n=1}^\infty$ is a sequence in $X$ with $\lim_{n\to\infty} x_n = 0$, prove the existence of a real sequence $\alpha_n \to \infty$ with $\alpha_n x_n \to 0$.
   (b) Show that a linear operator $T : X \to X$ is bounded if and only if the sequence $(Tx_n)_{n=1}^\infty$ is bounded whenever $x_n \to 0$.

2. Let $k(s,t)$ be a continuous function $[0,1] \times [0,1] \to \mathbb{R}$ and define an operator $K : L^1[0,1] \to C[0,1]$ by
   \[ Kf(s) = \int_0^1 k(s,t)f(t) \, dt. \]
   Prove that $K$ is compact.

3. Let $X$ be a Banach space and $M : X \to X$ an injective linear map such that both $M^2$ and $M^3$ are bounded operators on $X$. Prove that $M$ is bounded.

4. Let $H$ be a Hilbert space.
   (a) Prove that if a linear subspace $Y \subset H$ contains an open set, then $Y = H$.
   (b) Prove that if $H$ is infinite-dimensional, then a Hamel basis must consist of uncountably many elements.

5. Let $H$ be a Hilbert space and $T \in \mathcal{L}(H)$ a bounded linear operator.
   (a) Prove that if $T$ is symmetric then $T$ has a bounded inverse $T^{-1} \in \mathcal{L}(H)$ if and only if $T$ is bounded below, i.e., there is $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in H$.
   (b) Show that if $T$ is not symmetric, the statement in (a) might be false.

6. Define $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ by
   \[ Tf(x) = \frac{1}{1 + x^2} \int_x^\infty f(t) \, dt. \]
   Prove that $T$ is bounded and compute its norm $\|T\|$. 
