Problem 1. Denote by $X$ the subspace of $\ell^\infty$ consisting of all sequences $\{a_n\}_{n=0}^\infty$ in $\ell^\infty$ such that the limit $\lim_{n \to \infty} a_n$ exists. Set

$$\ell(\{a_n\}_{n=0}^\infty) = \lim_{n \to \infty} a_n, \quad \{a_n\}_{n=0}^\infty \in X.$$  

Clearly, the function $\ell$ is a linear functional on $X$ and the inequality $|\ell(a)| \leq \|a\|_\infty$ holds for $a \in X$. Existence of an extension $\Lambda$ of $\ell$ to the space $\ell^\infty$ (with $\|\Lambda\| \leq 1$) now follows by the Hahn-Banach theorem. We mention that such a continuous linear functional $\Lambda$ on $\ell^\infty$ is commonly called a Banach limit. \hfill $\square$

Problem 2. Denote by $\mathcal{H} = L^2((0, \infty), e^{-x}dx)$ the Hilbert space of all complex-valued measurable functions $f : (0, \infty) \to \mathbb{C}$ with finite norm

$$\|f\| = \left( \int_0^\infty |f(x)|^2 e^{-x}dx \right)^{1/2}.$$  

Denote by $F$ the set of all polynomials $p$ of degree at most 10 such that $p(1) = 2$, $p(2) = 3$ and $p(3) = 7$. Clearly $F$ is a non-empty closed convex subset of $\mathcal{H}$. A standard result now gives that the function $\log x$ in $\mathcal{H}$ has exactly one nearest point $p_0$ in $F$ (see Theorem 6.2 in Lax). \hfill $\square$

Problem 3. The adjoint Volterra operator $V^*$ acts as

$$V^* f(x) = \int_x^1 f(t)dt, \quad x \in (0, 1),$$  

on functions $f \in L^2(0, 1)$. \hfill $\square$

Problem 4. Notice first that $\|f_n\|_2 = \sqrt{2}$ for $n = 1, 2, 3, \ldots$. Observe also that

$$\langle f_n, \varphi \rangle = \frac{1}{\sqrt{n}} \int_{-n}^{n} \varphi(x)dx \to 0$$  

as $n \to \infty$ if $\varphi \in L^2(\mathbb{R})$ has compact support.

Let $f \in L^2(\mathbb{R})$. We show next that $\langle f_n, f \rangle \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ be given. Since compactly supported functions in $L^2(\mathbb{R})$ are dense in $L^2(\mathbb{R})$ we can find $\varphi$ in $L^2(\mathbb{R})$ compactly supported such that $\|f - \varphi\|_2 < \varepsilon/4$. Also since $\langle f_n, \varphi \rangle \to 0$ there is an integer $N$ such that $|\langle f_n, \varphi \rangle| < \varepsilon/2$ if $n \geq N$. For $n \geq N$ we now have

$$|\langle f_n, f \rangle| \leq |\langle f_n, f - \varphi \rangle| + |\langle f_n, \varphi \rangle| \leq \sqrt{2}\|f - \varphi\|_2 + |\langle f_n, \varphi \rangle| < \varepsilon$$  

by the triangle and Cauchy-Schwarz inequalities. This gives that $f_n \to 0$ weakly in $L^2(\mathbb{R})$. Since $\|f_n\| = \sqrt{2}$ for every $n$, the sequence $\{f_n\}$ is not norm convergent. \hfill $\square$

Problem 5. Consider for $x \in E$ the functional $\kappa x$ in the bidual $X^{**}$ of $X$ defined by

$$(\kappa x)(\ell) = \ell(x), \quad \ell \in X^*.$$  

Observe that $\sup_{x \in E}|(\kappa x)(\ell)| < +\infty$ for every $\ell \in X^*$ by assumption. An application of the Banach-Steinhaus theorem gives a finite positive constant $C$ such that

$$\sup_{x \in E}\|\kappa x\| \leq C.$$  

Anders Olofsson

Suggested Solutions. Functional Analysis. Exam 2010-12-18
Also $\|\kappa x\| = \|x\|$ by the Hahn-Banach theorem. We conclude that $\|x\| \leq C$ for every $x \in E$. \hfill \Box

**Problem 6.** From the course we know that $V$ is a compact bounded operator on $L^2(0,1)$ with spectrum $\sigma(V) = \{0\}$. If $V$ was normal we would have that $V = 0$ by the spectral theorem. Therefore $V$ is not normal. \hfill \Box

**Problem 6.** A more computational argument can be modeled as follows. By formula (1) above for $V^*$ we have that

\[ V + V^* = J, \]

where $J$ is the operator given by

\[ Jf(x) = \int_0^1 f(t)dt, \quad x \in (0,1), \]

for $f \in L^2(0,1)$. Multiplying from the left by $V$ we have $V^2 + VV^* = VJ$, and similarly that $V^2 + V^*V = JV$ by right multiplication by $V$. By these two identities we see that $V$ is normal if and only if $VJ = JV$. Straightforward calculation gives

\[ VJf(x) = x \int_0^1 f(t)dt, \quad x \in (0,1), \]

and similarly by a change of variables or integration by parts that

\[ JVf(x) = \int_0^1 (1-t)f(t)dt, \quad x \in (0,1), \]

for $f \in L^2(0,1)$. By these formulas it is evident that $VJ \neq JV$. We conclude that $V$ is not normal. \hfill \Box