Problem 1. Recall that the exponential monomials \( \{e^{in\theta}\}_{n=-\infty}^{\infty} \) form an orthonormal basis for \( L^2(\mathbb{T}) \). By Parseval’s formula the searched for minimizer is the function \( f = \sum_{n=1}^{\infty} \frac{1}{n} e^{in\theta} \) in \( L^2(\mathbb{T}) \). \( \square \)

Problem 2. Consider the linear map \( L : X \rightarrow \mathbb{C}^n \) defined by
\[
Lx = (\ell_1(x), \ldots, \ell_n(x)), \quad x \in X.
\]
Since the space \( X \) has infinite dimension, this map \( L \) has non-zero kernel
\[
\ker L = \{ x \in X : Lx = 0 \}.
\]
The subspace \( \ker L \) is unbounded and contained in \( U \). \( \square \)

Problem 3. Denote by \( Y \) the cartesian product \( Y = X_1 \times X_2 \) equipped with the norm
\[
\|(x_1, x_2)\| = \|x_1\| + \|x_2\|, \quad (x_1, x_2) \in Y,
\]
and coordinatewise vector operations. It is clear that \( Y \) is a Banach space. Consider the operator of addition \( A : Y \rightarrow X \) defined by
\[
A(x_1, x_2) = x_1 + x_2, \quad (x_1, x_2) \in Y.
\]
It is clear that \( A \in \mathcal{L}(Y, X) \) is a continuous linear operator. By assumption \( A \) is onto. By the open mapping theorem there exists a constant \( d > 0 \) such that \( B(0, d) \subset A(B(0, 1)) \), that is, every \( x \in X \) with \( \|x\| < d \) has a representation \( x = x_1 + x_2 \) with \( x_1 \in X_1, x_2 \in X_2 \) and \( \|x_1\| + \|x_2\| < 1 \). A scaling argument now gives that every \( x \in X \) has a representation \( x = x_1 + x_2 \) with \( x_1 \in X_1, x_2 \in X_2 \) and \( \|x_1\| + \|x_2\| \leq \|x\|/d' \) for every \( 0 < d' < d \). \( \square \)

Problem 4. Assume that \( x_n \rightarrow 0 \) weakly in \( \mathcal{H} \). We shall show that \( \|Bx_n\| \rightarrow 0 \). By compactness of \( A \) we have that \( \|Ax_n\| \rightarrow 0 \). Also, by the Banach-Steinhaus theorem, the sequence \( \{x_n\} \) is bounded: \( \|x_n\| \leq C \). Now
\[
\|Bx_n\|^2 = \langle Bx_n, Bx_n \rangle = \langle B^2 x_n, x_n \rangle = \langle Ax_n, x_n \rangle \leq \|Ax_n\||x_n| \rightarrow 0,
\]
where the inequality follows by Cauchy-Schwarz. Compactness of \( B = A^{1/2} \) now follows by a well-known criterion. \( \square \)

Problem 5. It is straightforward to see that every continuous linear functional \( \ell \) on the space \( c_0 \) has the form \( \ell = \ell_b \) for some \( b \in \{b_n\}_{n=0}^{\infty} \in \ell^1 \), where
\[
\ell_b(\{a_n\}_{n=0}^{\infty}) = \sum_{n=0}^{\infty} a_nb_n, \quad \{a_n\}_{n=0}^{\infty} \in c_0,
\]
and furthermore that
\[
\|\ell_b\| = \|b\|_{\ell^1} = \sum_{n=0}^{\infty} |b_n|.
\]
We mention that this fact also follows from a Riesz representation theorem describing the dual of continuous functions vanishing at infinity (see Rudin book Chapter 6). Since \( (\ell^1)^* = \ell^\infty \) by Riesz representation and \( \ell^\infty \neq c_0 \), we conclude that the space \( c_0 \) is not reflexive. \( \square \)
Problem 6. We notice first that $\Lambda_n(f) \to 0$ as $n \to \infty$ if $f$ is continuously differentiable with compact support in $(0, \infty)$ as follows by integration by parts. Indeed,

$$\Lambda_n(f) = \frac{1}{n} \int_0^{n\pi} \cos(nx)f'(x)dx$$

if $f$ is continuously differentiable with support in $[1/n, n\pi]$, and a passage to the limit gives that $\Lambda_n(f) \to 0$ as $n \to \infty$ if $f \in C_c(0, \infty)$. Since continuous functions with compact support in $(0, \infty)$ are dense in $L^p(0, \infty)$, the problem of weak$^*$ convergence of the sequence $\{\Lambda_n\}_{n=1}^\infty$ boils down to whether the sequence of norms $\|\Lambda_n\|$ stay bounded as $n \to \infty$.

By Riesz representation theorem we have that

$$\|\Lambda_n\| = \left( \int_0^{n\pi} |\sin(nx)|^q dx \right)^{1/q},$$

where $1/p + 1/q = 1$. A calculation gives

$$\|\Lambda_n\|^q = \int_0^{n\pi} |\sin(nx)|^q dx = \frac{1}{n} \int_0^{n^2\pi} |\sin(x)|^q dx = n \int_0^{\pi} |\sin(x)|^q dx \to \infty$$
as $n \to \infty$. By Banach-Steinhaus theorem, we conclude that the sequence $\{\Lambda_n\}_{n=1}^\infty$ has no weak$^*$ limit in the dual of $L^p(0, \infty)$. \qed