Solutions

1. The fixed points satisfy the system

\[
\begin{cases}
2 - y - x^2 = 0 \\
2x(x - y) = 0.
\end{cases}
\]

The second equation gives \( x = 0 \) or \( x = y \). If \( x = 0 \), the first equation gives \( y = 2 \). If \( x = y \), the first equation reduces to \( x^2 + x - 2 = (x + 2)(x - 1) = 0 \) with solutions \( x = -2 \) and \( x = 1 \). Hence, the fixed points are \((0, 2), (-2, -2)\) and \((1, 1)\).

The Jacobian of the right hand side is

\[
f'(x, y) = \begin{pmatrix} -2x & -1 \\ 4x - 2y & -2x \end{pmatrix}.
\]

Hence

\[
f'(0, 2) = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}
\]

with eigenvalues \( \pm 2 \),

\[
f'(-2, -2) = \begin{pmatrix} 4 & -1 \\ -4 & 4 \end{pmatrix}
\]

with eigenvalues 2 and 6, and

\[
f'(1, 1) = \begin{pmatrix} -2 & -1 \\ 2 & -2 \end{pmatrix}
\]

with eigenvalues \( -2 \pm \sqrt{2}i \).

It follows that \((1, 1)\) is asymptotically stable while \((0, 2)\) and \((-2, -2)\) are unstable.

2. Consider the initial value problem

\[
x' = 1 + \frac{x^2}{1 + t^2}, \quad x(0) = 0.
\]

a) We have

\[
|f(t, x) - f(t, y)| = \left| \frac{x^2 - y^2}{1 + t^2} \right| = \frac{|x + y|}{1 + t^2} \left| x - y \right| \leq 2|x - y|,
\]

when \((t, x), (t, y) \in [0, 1] \times [-1, 1]\). Hence, \( f \) is Lipschitz continuous with respect to \( x \) with Lipschitz constant 2.
b) We have that
\[ M = \sup_{(t,x) \in [0,1] \times [-1,1]} |f(t, x)| = 2. \]

Hence, the IVP has a unique solution on the interval \([0, T_0]\), where
\[ T_0 = \min \left\{ T, \frac{\delta}{M} \right\} = \min \left\{ 1, \frac{1}{2} \right\} = \frac{1}{2}, \]

by Picard-Lindelöf. In addition, the solution satisfies \(|x(t)| \leq \delta = 1\) there. To see that the solution is non-negative, we just note that \(x' \geq 0\).

3. The corresponding system is
\[
\begin{align*}
x' &= y, \\
y' &= -x^3 - y^3.
\end{align*}
\]
Making the Ansatz \(L(x, y) = ax^{2m} + by^{2n}\), we see that
\[ L(x, y) = x^4 + 2y^2 \]
is a Lyapunov function with
\[ \frac{d}{dt} L(x, y) = -4y^4 \leq 0 \]
along a solution. Moreover, the derivative is zero only when \(y = 0\). But then \(y' = -x^3 \neq 0\) if \(x \neq 0\). Hence the only orbit on which \(L\) is constant is the origin. It follows from Krasovskii-LaSalle’s theorem that the origin is an asymptotically stable fixed point.

4. The homogeneous equation \(y'' = 0\) has the general solution \(y(x) = a + bx\). The homogeneous boundary condition \(y(0) = 0\) gives \(a = 0\), while the boundary condition \(y'(1) = 0\) gives \(b = 0\). Hence, the homogeneous problem only has the trivial solution \(y(x) \equiv 0\). It follows that the problem in the exercise has a unique solution for each \(f\).
We can write the solution as \(y(x) = y_1(x) + y_2(x)\), where \(y_1\) solves the problem
\[
\begin{align*}
y''_1(x) &= f(x), & 0 < x < 1, \\
y_1(0) &= 0, \\
y'_1(1) &= 0,
\end{align*}
\]
and \(y_2\) solves the problem
\[
\begin{align*}
y''_2(x) &= 0, & 0 < x < 1, \\
y_2(0) &= 1, \\
y'_2(1) &= 2.
\end{align*}
\]
To compute Green’s function, we let \(u_1(x) = x\) be a non-trivial solution of the equation which satisfies the homogeneous boundary condition at \(x = 0\) and \(u_2(x) = 1\) a non-trivial solution satisfying the homogeneous boundary condition at \(x = 1\). The
corresponding Wronskian is \( W = u_1(x)u_2'(x) - u_1'(x)u_2(x) = -1 \). Hence, Green’s function is

\[
G(x, \xi) = \begin{cases} 
\frac{u_1(\xi)u_2(x)}{u_1(x)u_2(\xi)}, & 0 \leq \xi \leq x \leq 1 \\
\frac{u_1(x)u_2(\xi)}{u_1(x)u_2(\xi)}, & 0 \leq x \leq \xi \leq 1
\end{cases}
\]

and

\[
y_1(x) = \int_0^1 G(x, \xi)f(\xi)\,d\xi = -\int_0^x \xi f(\xi)\,d\xi - \int_x^1 x f(\xi)\,d\xi.
\]

On the other hand, \( y_2(x) = a + bx \) for some \( a \) and \( b \). Substituting this into the boundary conditions gives \( a = 1 \) and \( b = 2 \).

It follows that

\[
y(x) = 1 + 2x + \int_0^1 G(x, \xi)f(\xi)\,d\xi = 1 + 2x - \int_0^x \xi f(\xi)\,d\xi - \int_x^1 x f(\xi)\,d\xi.
\]

5. a) If \( \lambda = -k^2 < 0 \), we obtain that \( y(x) = a \sinh(kx) + b \cosh(kx) \). The boundary condition at \( x = 0 \) gives \( b = 0 \) and without loss of generality we can take \( y(x) = \sinh(kx) \). The boundary condition at \( x = 1 \) then reduces to \( k = \tanh k \). Setting \( f(k) = \tanh k - k \), we have that \( f(0) = 0 \) and \( f'(k) = \tanh^2 k > 0 \) for \( k \neq 0 \). Hence \( f(k) \neq 0 \) for \( k \neq 0 \). It follows that there are no negative eigenvalues.

If \( \lambda = 0 \), then \( y(x) = a + bx \) for some \( a \) and \( b \). The boundary conditions show that \( y \) is a solution iff \( a = 0 \). Hence, \( \lambda_0 = 0 \) is the first eigenvalue (with eigenfunction \( u_0(x) = x \), say).

If \( \lambda = k^2 > 0 \), we obtain that \( y(x) = a \sin(kx) + b \cos(kx) \) and from the boundary condition at \( x = 0 \) we get \( b = 0 \). We again take \( a = 1 \) so that \( y(x) = \sin(kx) \).

The boundary condition at \( x = 1 \) is then satisfied if and only if

\[
\tan k = k.
\]

It suffices to consider \( k > 0 \). The function \( \tan k \) is strictly increasing from \( -\infty \) to \( \infty \) on each interval \( ((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi) \), \( n \in \mathbb{Z} \). Moreover, in this interval it is positive when \( k \in (n\pi, (n + \frac{1}{2})\pi) \). Hence, there is at least one zero in each interval \( (n\pi, (n + \frac{1}{2})\pi) \). To show that there is precisely one, we note that

\[
\frac{d}{dk}(\tan k - k) = \tan^2 k > 0, \quad n\pi < k < (n + \frac{1}{2})\pi.
\]

It now follows that

\[
\lambda_1 \in \left(\pi^2, \left(\frac{3\pi}{2}\right)^2\right), \lambda_2 \in \left((2\pi)^2, \left(\frac{5\pi}{2}\right)^2\right), \ldots, \lambda_n \in \left((n\pi)^2, \left((n + \frac{1}{2})\pi\right)^2\right), \ldots
\]

Please, turn over!
b) Let $k_n = \sqrt{\lambda_n}$ and $\delta_n := (n + \frac{1}{2}) \pi - k_n \in (0, \frac{\pi}{2})$, $n \geq 1$. Note that $\tan k_n = k_n$ is equivalent to $\frac{1}{k_n} = \cot k_n$.

We have

$$\cot k_n = \cot ((n + \frac{1}{2}) \pi - \delta_n) = \cot \left( \frac{\pi}{2} - \delta_n \right) = \tan \delta_n = \frac{1}{(n + \frac{1}{2}) \pi - \delta_n} \to 0$$

as $n \to \infty$. Hence $\delta_n \to 0$ as $n \to \infty$ (this is also easy to see geometrically). By Taylor expanding, we find that

$$\frac{1}{(n + \frac{1}{2}) \pi - \delta_n} = \tan \delta_n = \delta_n + O(\delta_n^3).$$

Hence,

$$\delta_n (1 + O(\delta_n^2)) = \frac{1}{(n + \frac{1}{2}) \pi} \frac{1}{1 + O(\delta_n)}$$

and

$$\left( n + \frac{1}{2} \right) \pi \delta_n = \frac{1}{1 + O(\delta_n)} \to 1$$

as $n \to \infty$. But this means that

$$\lambda_n - \left( (n + \frac{1}{2}) \pi \right)^2 = k_n^2 - \left( (n + \frac{1}{2}) \pi \right)^2$$

$$= \left( (n + \frac{1}{2}) \pi - \delta_n \right)^2 - \left( (n + \frac{1}{2}) \pi \right)^2$$

$$= \left( (n + \frac{1}{2}) \pi \right)^2 - 2 \left( n + \frac{1}{2} \right) \pi \delta_n + \delta_n^2 - \left( (n + \frac{1}{2}) \pi \right)^2$$

$$= -2 \left( n + \frac{1}{2} \right) \pi \delta_n + \delta_n^2 \to -2$$

as $n \to \infty$. 