Students may use the formula sheet and a pocket calculator. All solutions should be properly justified.

1. Consider the function \( f(x, y) = x^2 - 2x - xy + e^{xy} \) in \( \mathbb{R}^2 \).

   a) Show that the global minimum exists and find it. (0.4)

   b) Do, if possible, one Newton step from the point \((0, 1)\). Explain. (0.3)

   c) Do one Modified Newton step. Is the search direction descent? Do you get a smaller functional value? Explain. (0.3)

2. a) Prove using Farkas theorem that

   \[
   \begin{align*}
   2x - y &\leq 0, \\
   -x + 3y &\leq 0 \\
   \Rightarrow & \\
   3x + 5y &\leq 0.
   \end{align*}
   \] (0.5)

   b) Replace \( 3x + 5y \leq 0 \) in 2a) with \( c_1 x + c_2 y \leq 0 \) and plot all possible vectors \( c = (c_1 \ c_2) \) in the plane such that the implication still holds. (0.5)

3. a) Solve the LP problem using Simplex algorithm (without Phase I)

   \[
   \begin{align*}
   \max (2x_1 + 8x_2 + 6x_3) \quad \text{subject to} \quad & x_1 + 3x_2 + x_3 \leq 6, \\
   & 3x_1 + 5x_2 + x_3 \leq 7, \\
   & 3x_1 + x_2 + x_3 \geq 2, \\
   & \text{all } x_k \geq 0
   \end{align*}
   \] (0.5)

   b) State the dual problem to 3a) and solve it by Complementary Slackness Principle. (0.5)

Please, turn over
4. a) Prove that if for \( i = 1, 2, \ldots, m \) the functions \( g_i \) are convex then the set 
\[ S = \{ x \in \mathbb{R}^n : g_i(x) \leq \alpha_i, i = 1, 2, \ldots, m \} \] 
is convex for any \( \alpha_i \in \mathbb{R} \). (0.3)

b) Prove that if the functions \( f \) and \( g_i, i = 1, 2, \ldots, m, \) are convex then 
for any constant \( \epsilon > 0 \) the barrier method function
\[
F(x) = f(x) + \epsilon \sum_{i=1}^{m} \frac{-1}{g_i(x)}
\]
is convex on \( S \) defined in 4a) when all \( \alpha_i = 0 \). (0.4)

c) Find all \( a \in \mathbb{R} \) such that the function \( f(x, y) = x^2 + 2xy^2 + ay^4 \) is 
convex in \( \{(x, y) : x \geq 0\} \). (0.3)

5. Solve the optimization problem
\[
\min (x^2 - 2xy + 2y) \quad \text{subject to} \quad y^2 \leq 2x, \ y \geq 0
\]
using the KKT necessary condition.

6. a) State and prove the theorem that claims the equivalence of the existence of a 
saddle point and no duality gap. (0.4)

b) Solve the problem in 5 by duality method: take \( X = \{y \geq 0\} \), calculate the dual 
function, solve the dual problem and prove no duality gap. (0.6)

GOOD LUCK!