Students may use the formulae sheet and a pocket calculator. All solutions should be properly justified.

1. Decide whether the statement is correct or not and provide a short explanation.
   a) The Golden Section method always converges to a local minimum on the initial interval. (0.2)
   b) The Newton method may diverge even for a quadratic function. (0.2)
   c) Two successive directions in the Steepest Descent method are orthogonal. (0.2)
   d) The exact line search gives a point at which the search direction is tangent to the level set of the minimized function. (0.2)
   e) If $a$ is a global minimum for a constrained convex problem then the gradient at $a$ of the minimized function is zero. (0.2)

2. a) Is the matrix

   $\begin{pmatrix}
   1 & 1 & -2 \\
   1 & 3 & -4 \\
   -2 & -4 & 9
   \end{pmatrix}$

   positive-definite? (0.2)
   b) Is the function $f(x, y) = x^4 + 2x^2y^2 + y^4$ convex on $\mathbb{R}^2$? (0.4)
   c) Prove that the function $f(x, y) = e^{x^2+y^2} + \ln(e^{x+y} + 1)$ is convex on $\mathbb{R}^2$. (0.4)

3. Let the set $S \subset \mathbb{R}^2$ be described by inequalities

   $S : \begin{cases}
   x_1 - x_2 \geq 0, \\
   -x_1 + 2x_2 \geq 2, \\
   x_2 \leq 4, \\
   x_1, x_2 \geq 0.
   \end{cases}$

   a) Convert the set of inequalities into the canonical form and use Simplex Phase 1 to find a feasible point. (0.5)
   b) Using the LP duality and the CSP$^1$, show that (4, 4) is the optimal point for the problem: $\min(x_1 - 3x_2)$ subject to $(x_1, x_2) \in S$. (0.5)

Please, turn over

$^1$Complementary Slackness Principle
4. a) Give the definition of the convex hull of a set $S \subset \mathbb{R}^n$.  

b) Solve graphically the optimization problem

$$\min ax - y \quad \text{subject to } y \leq x^2, \ 0 \leq x \leq 1, \ y \geq 0$$

for all possible values of the real parameter $a$.  

(0.2)

c) Let $S \subset \mathbb{R}^n$, $c \in \mathbb{R}^n$ and let $f(x) = c^T x$ be a linear function. Denote by $H(S)$ the convex hull of the set $S$. Prove that

$$\min_{x \in S} f(x) = \min_{x \in H(S)} f(x).$$

(Hint: Prove $\geq$ and $\leq$ separately.)  

(0.3)

5. Consider the optimization problem

$$\min (y^2 - 3x) \quad \text{subject to } y \geq x^3, \ y \geq 0.$$  

a) Solve the problem by the KKT-CQ necessary condition.  

b) Replace the condition $y \geq x^3$ with the equivalent $\sqrt[3]{y} \geq x$ and explain how the problem can be solved without knowing that the minimum exists.  

(0.6)

(0.4)

6. a) Consider the same problem as in 5b), that is

$$\min (y^2 - 3x) \quad \text{subject to } \sqrt[3]{y} \geq x, \ y \geq 0.$$  

Define $X = \{(x, y) \mid y \geq 0\}$. Calculate the dual function, solve the dual problem and check the duality gap.  

b) Let the Conjugate Direction method in combination with an exact line search be applied to a quadratic polynomial with a positive definite Hessian. Prove that the multidimensional search ends after a finite number of steps.  

(0.5)

(0.5)

GOOD LUCK!