1. Consider the set $S = \{(x, y) \in \mathbb{R}^2: y^2 \leq x \leq y\}$.
   a) Draw the set $S$ and the set of all feasible directions at $(1, 1)$. (0.2)
   b) Assume that $(1, 1)$ is the local minimum point for some function $f(x, y)$ in $S$. Draw the set of all possible gradients $\nabla f(1, 1)$. Which of the eight vectors $(\pm 3, \pm 2)$ and $(\pm 2, \pm 3)$ belong to this set? (0.5)
   c) Does the optimization problem
      \[
      \text{minimize } e^{(x-y)^2} - \cos(\ln(x^4 + y^{2014})) \quad \text{subject to } (x, y) \in S
      \]
      have a solution? (0.3)

2. Consider the matrix $A$ and the (column) vector $b$ given by
   \[
   A = \begin{pmatrix}
   1 & 0 & 1 & -2 \\
   -2 & 2 & 1 & -1 \\
   1 & 1 & -1 & 2
   \end{pmatrix}, \quad b = \begin{pmatrix}
   1 \\
   6 \\
   1
   \end{pmatrix}.
   \]
   a) Find out whether $b$ belongs to the convex hull of the columns of $A$. (0.4)
   b) Suggest three different methods (analytical or numerical) that you think may be applied to solve the optimization problem
      \[
      \text{min } \|Ax - b\|^2 \quad \text{subject to } x \geq 0.
      \]
      Provide details and explanations why your choice is going to work. (0.6)

3. Consider the matrix $A$ and the vector $b$ from the Problem 2.
   a) Use the Simplex method (Phase 1) to find out whether the set
      \[
      S = \{x \in \mathbb{R}^4: Ax = b, \ x \geq 0\}
      \]
      is nonempty. In that case, calculate a basic feasible solution. (0.5)
   b) Consider the problem of minimizing $f(x_1, x_2, x_3, x_4) = x_1 - x_4$ subject to $Ax = b$ and $x \geq 0$. Find out by the Complementary Slackness Principle whether the vector $(1, 0, 16, 8)$ is the optimal solution. (0.5)

Please, turn over
4. a) Prove that the sum of two convex functions is a convex function. (0.4)

b) Denote \( f(x, y, z) = e^{x+y+z} - 1 \). Are the following functions convex in \( \mathbb{R}^3 \)

- \( g(x, y, z) = |\max\{f(x, y, z), 0\}| \)? (0.3)
- \( h(x, y, z) = \max\{|f(x, y, z)|, 0\}| \)? (0.3)

Motivate your answer!

5. Solve the following problem by KKT method

\[
\min xyz \quad \text{subject to} \quad x^2 + y^2 + z \leq 4, \ z \geq 0.
\]

6. a) Solve the following optimization problem

\[
\min x^2 + y^2 + xy^2 + y \quad \text{subject to} \quad 4 \leq y^2 - x^2, \ x \geq 0.
\]

using the duality method with the set \( X = \{(x, y) \in \mathbb{R}^2 : x \geq 0\} \). (0.5)

b) Prove the existence and uniqueness of the global minimum for a quadratic polynomial with a positive-definite Hessian. (0.5)

GOOD LUCK!