1. a) Give the definition of a feasible direction for a set $S$ at a point $a \in S$. 
   
   b) Consider the set $S = \{(x, y) \in \mathbb{R}^2: 0 \leq x \leq \pi, 0 \leq y \leq \sin x\}$. Plot the set of all feasible directions for $S$ at $(0, 0)$ as well as the set of all possible gradients of a function that satisfy the necessary condition for a local minimum at $(0, 0)$. Make the axes scaling clear to understand.
   
   c) Find all $a \in \mathbb{R}$ such that the vector $(a, 1, a)$ belongs to the convex hull of the following vectors $(1, 1, 1), (1, 2, 2), (2, 1, 2), (0, 0, 0)$.

2. a) Give the definition of a descent direction for a function $f$ at a point $a$. 
   
   b) Find out if Newton’s direction at $(1, 0)$ is a descent one for the following function $f(x, y) = x^4 - 2x^2y + y^2e^y$.
   
   c) Verify for the function $f$ in 2b) that $\min_{x \in \mathbb{R}^2} f(x)$ exists and calculate it.

3. a) Use the Complementary Slackness Principle to show that the solution to the following LP problem
   
   \[
   \begin{align*}
   \max (x_1 + 4x_2 + 3x_3) \quad \text{subject to} \quad & \begin{cases} 
   x_1 + 2x_2 + 2x_3 \leq 5, \\
   2x_1 + 2x_2 + 4x_3 \geq 2, \\
   3x_1 + x_2 - x_3 \leq 3, \\
   \text{all } x_k \geq 0.
   \end{cases}
   \end{align*}
   \]
   
   satisfies $x_1 = x_3 = 0$ and solve the problem.
   
   b) Use Farkas’ theorem to prove that
   
   \[
   \begin{cases} 
   x_1 + 2x_2 + x_3 \leq 0, \\
   x_1 + x_2 + 2x_3 \leq 0, \\
   2x_1 - x_2 + x_3 \leq 0.
   \end{cases} \Rightarrow 5x_1 + 3x_3 \leq 0.
   \]
4. a) Prove that if the function \( f \) is convex on \( S \) then the set
\[
A = \{ x \in S : f(x) \leq \alpha \}
\]
is convex for any \( \alpha \in \mathbb{R} \). (0.3)
b) Consider the matrix
\[
H = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.
\]
Is the set
\[
\Omega = \{ x \in \mathbb{R}^3 : x^T H x \leq 1 \}
\]
convex? (0.3)
c) Is the set \( \{ (x, y) \in \mathbb{R}^2 : |\ln(x + y)| \leq 1, x \geq 0, y \geq 0 \} \) convex? (0.4)

5. Solve the following optimization problem by the KKT method
\[
\min(y^2 - 2xz) \text{ subject to } x + yz + z^3 \leq 2, z \geq 0.
\]

6. a) Solve the optimization problem
\[
\min(x + y + z) \text{ subject to } x^2 + y^2 \leq z, z \leq 1
\]
by the duality method with \( X = \{ (x, y, z) : z \leq 1 \} \). (0.5)
b) Prove the existence and uniqueness of the global minimum for a quadratic polynomial with the positive-definite Hessian. (0.5)

GOOD LUCK!