Answers and Comments

1. a) The cone of feasible directions is \( \{(x, y) : x \leq y < \frac{1}{2}x\} \) (def. on p. 235).
   
   b) The cone of possible gradients is \( \{(x, y) : y \leq -x, y \leq -2x\} \).
      
   \((-2, -3)\) and \((-3, \pm 2)\) are possible gradients (Lemma 1, p. 235).
   
   c) Set \( y = x \) to minimize the first term. Then within the set \( 0 \leq x^4 + x^{2014} \leq 2 \).
      
   Clearly there exists \( 0 \leq x_0 \leq 1 \) such that \( x_0^4 + x_0^{2014} = 1 \) which minimizes the second term. So the minimum point \((x_0, x_0) \in S\) exists (and min=0).

2. a) No. The second equation in the convex combination \(-2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 = 6\) is impossible since \(-2\lambda_1 + 2\lambda_2 + \lambda_3 - \lambda_4 \leq 2\lambda_2 + \lambda_3 < 2 + 1 = 3 < 6\).
   
   b) It is constrained nonlinear optimization due to \( x \geq 0 \). Analytical methods: KKT, duality (no gap by convexity), and numerical methods: penalty, barrier, are possible. One more trick is to use 3a) that gives min \( \|Ax - b\| = 0 \).

3. a) The set is nonempty. A basic feasible solution is \((0, 2, 3, 1)^T\).
   
   b) Set \( c = (1, 0, 0, -1)^T \). The dual problem is max \( b^T y \) subject to \( A^T y \leq c \). The CSP gives equality in the 1st, 3rd and 4th rows. Solve it: \( y = (0, -1, -1)^T \). The inequality in the 2nd row is OK, hence, a feasible point for the dual. Finally \( b^T y = c^T x = -7 \) shows no duality gap, hence, the optimal points.

4. a) It follows immediately from the definition of convex function that for \( h = f + g \)
   
   \[ h(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \lambda g(x) + (1-\lambda)g(y) = \lambda h(x) + (1-\lambda)h(y), \]
   
   which proves convexity of \( h \).
   
   b) By Lemma 2(2), p. 211, \( t = \max\{f, 0\} \geq 0 \) convex \( \Rightarrow g = |t| = t \) convex too.
      
   \( h = \max\{|f|, 0\} = |f| \). With \( y = z = 0 \) the function \( |f(x, 0, 0)| = |e^x - 1| \) is not convex (e.g. can be seen by the graph) \( \Rightarrow h \) is not convex.

5. The set is compact \( \Rightarrow \) min exists. No CQ points. KKT points: many stationary and other KKT points that give \( f = 0 \) and two KKT points \((1, -1, 2), (-1, 1, 2)\) that give \( f = -2 \). The last two are the minimum points.

6. a) The dual function is \( \Theta(u) = 4u - \frac{1}{4(1-u)} \) for \( 0 \leq u < 1 \) (otherwise \( -\infty \)). The optimal \( \tilde{u} = 3/4 \) and \( \Theta(3/4) = 2 \). Since \( f(0, -2) = 2 = \Theta(3/4) \), there is no duality gap \( \Rightarrow \) the minimum point is \((0, -2)\).
   
   b) See the book, Ex 7, p. 16 (plus Th. 13, p. 216 and Corollary 1, p. 215).

\(^1\)For re-exams only answers are provided.