1. Let’s calculate the gradient and the Hessian at the point \((x, y, z) = (0, 1, 3)\)

\[
\nabla f = \begin{bmatrix}
-x \\
(y^2 - z) \cdot 2y \\
z - y^2
\end{bmatrix} = \begin{bmatrix}
0 \\
-4 \\
2
\end{bmatrix},
\quad H = \begin{bmatrix}
1 & 0 & 0 \\
0 & 6y^2 - 2z & -2y \\
0 & -2y & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -2 \\
0 & -2 & 1
\end{bmatrix}.
\]

a) It is necessary for positive-semidefiniteness that \(\det(H_k) \geq 0\) (see page 360 after Example 5). Since \(\det(H) = -4 \not< 0\), \(H\) is not positive-semidefinite (and then not positive-definite either). Hence, the function cannot be convex (Theorem 13, page 216).

b) \[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} - H^{-1}\nabla f = \begin{bmatrix}
0 \\
1 \\
3
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/4 & -1/2 \\
0 & -1/2 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
-4 \\
2
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}.
\]

Since \(f \geq 0\) everywhere and \(f(0, 1, 1) = 0\), we have arrived to a global minimum. It is easy to see that there are many global minima here: \((0, t, t^2), t \in \mathbb{R}\).

2. Define \(f(x, y) = x - y, g_1(x, y) = -y, g_2(x, y) = y - x^3\), and calculate

\[
\nabla f = \begin{bmatrix}
1 \\
-1
\end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix}
0 \\
-1
\end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix}
-3x^2 \\
1
\end{bmatrix}.
\]

a) Draw the set \(g_1 \leq 0\) and \(g_2 \leq 0\). We get

- \((0, 0)\) is a CQ point as \(\nabla g_1\) and \(\nabla g_2\) have the opposite directions.
- \((a, a^3)\) is a KKT point as \(\nabla f\) and \(\nabla g_2\) have the opposite directions.

To find \(a > 0\): solve

\[
\det[\nabla f \ \nabla g_2] = \det\begin{bmatrix}
1 & -3a^2 \\
-1 & 1
\end{bmatrix} = 0 \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}}.
\]

b) The CQ point is a local minimum (feasible points are locally from one side of the level curve), the KKT point is a saddle point (feasible points are locally from both sides of the level curve). The global minimum does not exist (one can move the level curve against \(\nabla f\) up to infinity without loosing contact with feasible points).
3. a) A form of the dual LP problem is

\[
\begin{align*}
\text{min } (2y_1 + y_2 - y_3) & \text{ subject to } \\
y_2 + y_3 & \geq 0, \\
y_1 - 3y_2 + y_3 & \geq 1, \\
y_1 + 2y_2 - y_3 & \geq 2, \\
y_1 & \geq 0, \quad y_2 \leq 0, \quad y_3 \text{ free.}
\end{align*}
\]

CSP gives:

- the primal slack \( s_2 = -4 \neq 0 \Rightarrow y_2 = 0, \\
- the primal \( x_1 = 1 \neq 0 \Rightarrow y_2 + y_3 = 0 \Rightarrow y_3 = 0, \\
- the primal \( x_3 = 2 \neq 0 \Rightarrow y_1 + 2y_2 - y_3 = 2 \Rightarrow y_1 = 2. \\

It gives the dual feasible solution \( y = (2, 0, 0) \) and min = 4 = max from 3a, hence, both are optimal.

b) We rewrite the first system to get one of the Farkas canonical forms

\[
\begin{align*}
\begin{bmatrix} A \\ B \\ -B \\ c^T \end{bmatrix} x & \leq 0, \\
c^T x & > 0.
\end{align*}
\]

Then Farkas’ theorem gives us the alternative system as

\[
\begin{align*}
\begin{bmatrix} A^T & B^T & -B^T \end{bmatrix} \begin{bmatrix} y \\ v \\ w \end{bmatrix} & = c, \\
y, v, w & \geq 0.
\end{align*}
\]

Now introduce the new variable \( z = v - w \), which is no longer sign-definite.

4. a) It is the least square optimization. To solve it we will solve the normal equation \( A^T Ax = A^T b \) which in this case looks like

\[
\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} x = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \iff x = \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix}.
\]

b) We have

\[
\|Ax - b\|^2 = 3x_1^2 + 2x_2^2 - 12x_1 + 2x_2 + 14.
\]

Let us denote \( x_1 = x, x_2 = y \). So we need to solve the problem

\[
\text{min}(3x^2 + 2y^2 - 12x + 2y) \quad \text{subject to } x \geq 0, \quad y \geq 0.
\]

The problem is convex (prove it!), hence it is sufficient to find a KKT point. The KKT system is

\[
\begin{align*}
6x - 12 - u_1 & = 0, \\
4y + 2 - u_2 & = 0, \\
u_1 x & = 0, \\
u_2 y & = 0, \\
x \geq 0, \quad y \geq 0.
\end{align*}
\]

One solution is, for example, \((x, y) = (2, 0)\) with \( u_1 = 0, u_2 = 2. \)

Another (longer) solution: Use the KKT/CQ necessary condition.
5. a) Denote $\gamma = \min_{x \in S} f(x)$. The set of all solutions can be described as
\[ \{ x \in S : f(x) \leq \gamma \} \]
which is convex (prove it similar to Theorem 7, page 210).

Another solution: Prove convexity by definition.

b) Take two points $P = (0, 0, 0)$ and $Q = (0, 1, 1)$. We have $f(P) = f(Q) = 0$ and
\[ f(P/2 + Q/2) = f(0, 1/2, 1/2) = \frac{1}{4} > 0 = f(P)/2 + f(Q)/2. \]
Hence, the function is not convex as it does not satisfy the definition for $P$, $Q$ and $\lambda = 1/2$.

c) Define the set $S = \{ x > 0, y > 0, z > 0 \}$. It is convex as an intersection of three half-spaces. The function $f(t) = \frac{1}{t}$ is convex for $t > 0$ (as $f'' > 0$). The function $g(x, y, z) = x + y$ is affine and takes only positive values on $S$. Therefore, the superposition
\[ f(g(x, y, z)) = \frac{1}{x + y} \]
is convex by Lemma 2, page 211. Similarly, two other functions are convex.

Finally, the sum of convex functions is convex, and the convex function less than or equal to 1 on $S$ defines, thus, a convex set.

6. a) Let us denote $x_1 = x$, $x_2 = y$ again. As $y \geq 0$ is taken as an implicit constraint, the Lagrange function is
\[ L(x, y, u) = 3x^2 + 2y^2 - 12x + 2y - ux = 3x^2 - (12 + u)x + 2y^2 + 2y \quad \geq 0. \]
The expression $2y^2 + 2y$ is clearly non-negative on $X = \{ y \geq 0 \}$, then the minimum of $L$ with respect to $y \geq 0$ is attained at $y = 0$. To minimize $L$ further with respect to $x \in \mathbb{R}$ we notice that it is a convex function in $x$, and the minimum is attained at the stationary point
\[ 6x - 12 - u = 0 \quad \Rightarrow \quad x = \frac{12 + u}{6} = 2 + \frac{u}{6}. \]
Substitution of this $x$ and $y = 0$ to $L$ gives the minimum as the following dual function
\[ \Theta(u) = -12 - 2u - \frac{u^2}{12}, \quad u \geq 0. \]
Since $\Theta' = -2 - \frac{u}{6} < 0$ for $u \geq 0$, the dual function is decreasing, thus, the maximum is attained at $\bar{u} = 0$ where $\Theta(\bar{u}) = -12$. Taking the candidate $(\bar{x}, \bar{y})$ from the minimization of $L$ above
\[ (\bar{x}, \bar{y}) = (2 + \bar{u}/6, 0) = (2, 0) \]
we get the corresponding value of $f(\bar{x}, \bar{y}) = 12 - 24 = -12$ to be equal to $\Theta(\bar{u})$. It means that we have no duality gap, and the point $(2, 0)$ is the global minimum.

b) See Theorem 5 on page 264 and Corollary 1 on page 265.