Brief comments and answers only.

1. a) See Lemma 2, p. 121.

b) Similar to Exercise 4.12. We have

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & -1 & 2 \\
0 & 1 & 0 & 1 \\
0 & 2 & 2 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{bmatrix}
= 1
\iff
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4
\end{bmatrix}
= \begin{bmatrix}
5 - 6a \\
3 - 3a \\
5a - 4 \\
4a - 3
\end{bmatrix}.
\]

Now all \( \lambda_k \geq 0 \) is equivalent to \( \frac{1}{4} \leq a \leq \frac{5}{6} \).

2. a) \( q_\mu(x) = (x_2 + 5)^2 + \mu \max(0, 2 - 2x_1 - x_2)^2 + \mu \max(0, x_1 - x_2 - 4)^2 \) Convex as a sum of three convex functions. The last two are superpositions of the convex \( p(t) = \max(0, t)^2 \) (draw the graph to see it) and affine ones.

b) Around \((0, 0)\) we have \( q_{10}(x) = (x_2 + 5)^2 + 10(2 - 2x_1 - x_2)^2 \), thus \( \nabla q_{10}(0) = \begin{bmatrix} -80 \\ -30 \end{bmatrix} \), \( \nabla^2 q_{10}(0) = \begin{bmatrix} 80 & 40 \\ 40 & 22 \end{bmatrix} \) and \( d_{SD} = \begin{bmatrix} 80 \\ 30 \end{bmatrix} \uparrow \begin{bmatrix} 8 \\ 3 \end{bmatrix} \), \( d_N = \frac{1}{2} \begin{bmatrix} 7 \\ -10 \end{bmatrix} \). Newton’s direction points closer to the minimum (draw the set of constraints and the level sets of \( f(x) = (x_2 + 5)^2 \) to see that the minimum is \((2, -2)\)).

3. a) Use Farkas theorem to conclude that there exists \( y \) such that

\[
\begin{cases}
y_1 + 2y_2 = a, \\
y_1 + y_2 = 1, \\
y_2 \geq 0
\end{cases}
\iff
\begin{cases}
y_1 = \frac{a-2}{3} \geq 0, \\
y_2 = \frac{a+1}{3} \geq 0
\end{cases}
\iff a \geq 2.
\]

b) Primal: \( \min \begin{bmatrix} 0 & 1 \end{bmatrix} x \) subject to \( \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} x \geq \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \), \( x_1 \geq 0 \), \( x_2 \) free.

Dual: \( \max \begin{bmatrix} 2 & -4 & -3 \end{bmatrix} y \) subject to \( \begin{cases}
2y_1 - y_2 \leq 0, \\
y_1 + y_2 - y_3 = 1, \\
y_3 \geq 0.
\end{cases} \)

Testing \((0, 2)\) by CSP: primal feasible, the second and the third inequalities are strict, hence, \( y_2 = y_3 = 0 \). Moreover, \( y_1 = 1 \). However, \((1, 0, 0)\) is not dual feasible as the first dual inequality is false. Hence, \((0, 2)\) is not the optimal point. (It is easy to see graphically that the optimal solution is \((2, -2)\).)

4. The set is compact, thus, the minimum exists by Weierstrass theorem.

CQ condition is violated at the CQ points \((0, \pm 1, 0)\). KKT points are \((0, \pm 1, 0)\) (the same as CQ points) and \( \pm (1, 0, -1) \). The minimum is \(-2\) at \( \pm (1, 0, -1) \).
5. a) The Lagrangian is \( L(x, u_1, u_2) = x_1(u_2 - 2u_1) + x_2(1 - u_1 - u_2) + 2u_1 - 4u_2 \).

To minimize \( x_1(u_2 - 2u_1) \) over \( x_1 \geq 0 \): if \( u_2 - 2u_1 \geq 0 \) then the minimum is at \( x_1 = 0 \), otherwise the "minimum" is \(-\infty\). Similarly, to minimize \( x_2(1 - u_1 - u_2) \) over \( x_2 \leq 3 \): if \( 1 - u_1 - u_2 \leq 0 \) then the minimum is at \( x_2 = 3 \), otherwise the "minimum" is \(-\infty\). It gives

\[
\Theta(u_1, u_2) = \begin{cases} 
3 - u_1 - 7u_2, & \text{if } u_2 \geq 2u_1, u_1 + u_2 \geq 1, \\
-\infty, & \text{otherwise}.
\end{cases}
\]

Graphical minimization gives easily the optimal \( \tilde{u} = \left( \frac{1}{3}, \frac{2}{3} \right) \) with the optimal value \(-2\). (Comparison with the primal value at \((0,2)\) in 3b makes another proof that \((0,2)\) is not the optimal solution — it must be no duality gap for the LP, that is the optimal \( x_2 = -2 \).)

b) Denote \( w = \begin{bmatrix} u \\ v \end{bmatrix} \) and \( G(x) = \begin{bmatrix} g(x) \\ h(x) \end{bmatrix}^T \). Then \( L(x, w) = f(x) + G(x)w \) and \( \Theta(w) = \inf_{x \in X} L(x, w) \). We have for any \( x \in X \), and \( 0 \leq \lambda \leq 1 \) that

\[
L(x, \lambda w_1 + (1 - \lambda) w_2) = \lambda L(x, w_1) + (1 - \lambda) L(x, w_2) \geq \lambda \Theta(w_1) + (1 - \lambda) \Theta(w_2).
\]

Taking the infimum over \( x \in X \) we get

\[
\Theta(\lambda w_1 + (1 - \lambda) w_2) \geq \lambda \Theta(w_1) + (1 - \lambda) \Theta(w_2),
\]

thus, \( \Theta \) is concave by definition.

6. a) We have

\[
\nabla_{xx}^2 L = \begin{bmatrix} u & 0 & 1 \\
0 & u + v - 1 & 0 \\
1 & 0 & v \end{bmatrix}.
\]

At \((0, \pm 1, 0)\): \( u = 0, v = 1 \) (from KKT condition in 4) and the admissible \( d = (d_1, d_2, d_3) \neq 0 \) are \( \{d: d_2 = 0\} \). The matrix is indefinite on this subspace, hence, \((0, \pm 1, 0)\) are not strict local minima.

At \( \pm (1, 0, -1)\): \( u = v = 1 \) and admissible \( d \) satisfies \( d_3 = 0 \). The matrix is positive-definite on this subspace, hence, \( \pm (1, 0, -1) \) are strict local minima.

b) Primal: \( \min c^T x \mid Ax = b, x \geq 0 \), dual: \( \max b^T y \mid A^T y \leq b \). We have

\[
\exists x: \begin{cases}
Ax = b, \\
c^T x = \alpha,
\end{cases} \iff \exists x: \begin{cases}
[ A ] [ x ] = [ b ] \\
c^T x \geq 0
\end{cases}, \iff \text{[Farkas]}
\]

\[
\iff \mathcal{B}(y, w): \begin{bmatrix} A^T \\ c \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} \leq 0, \iff \mathcal{B}(y, w): \begin{bmatrix} A^T y \leq -cw, \\
b^T y > -\alpha w. \end{bmatrix}
\]

In particular, if we set \( w = -1 \) we can see that

\[
\mathcal{B} y: \begin{cases}
A^T y \leq c, \\
b^T y > \alpha,
\end{cases} \text{ or, equivalently, } A^T y \leq c \Rightarrow b^T y \leq \alpha.
\]