Throughout the exam, all functions are assumed to be real-valued.

Note: Only students who are registered or re-registered on the course are allowed to take the exam.

Test results: Posted Wednesday, June 1, before 17.00. Official viewing of the marked scripts: Thursday, June 2, 11.30-12.00, in room 563.

Oral exams: Wednesday, June 8 – Friday, June 10. State your preference (day and AM/PM) on the cover sheet of your test – at least two options.

1. a) Find a $C^2$ solution of the heat equation

$$u_t = u_{xx}$$

for $t > 0$ and $x \in (0, \pi)$, with mixed boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0$$

and initial condition

$$u(x,0) = \sin \left( \frac{x}{2} \right).$$

b) Show that this is the only solution in $C^2([0, \pi] \times [0, \infty)).$

2. Solve the problem

$$xu_x + u_y = 2yu, \quad y > 0,$$

with $u(x,0) = g(x)$, where $g \in C^1(\mathbb{R})$ is a given function. Verify that your answer really is a solution to the problem.

3. Let $g \in \mathcal{S}(\mathbb{R}^n)$ and recall that there is a unique bounded smooth solution of the heat equation $u_t = \Delta u$ in $\mathbb{R}^n \times (0, \infty)$ with $u(x,0) = g(x)$, $x \in \mathbb{R}^n$, and that $u(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)$ for each $t \geq 0$. Suppose that $\phi \in C^2(\mathbb{R})$ is convex with $\phi(0) = 0$. Show that

$$\int_{\mathbb{R}^n} \phi(u(x,t)) \, dx$$

is a decreasing function of $t$, where decreasing is used in the sense $f(t) \leq f(s)$ if $s \leq t$. Use this to show that the $L^p$ norm $\|u(\cdot, t)\|_p = (\int_{\mathbb{R}^n} |u(x,t)|^p \, dx)^{1/p}$ is a decreasing function of $t$ for each $p \in [2, \infty)$.

Remark: One can show that the $L^p$ norm is decreasing also for $p \in [1, \infty)$ by approximating $|u|^p$ with the smooth function $(u^2 + \varepsilon^2)^{p/2} - \varepsilon^p$, $\varepsilon > 0$, but this is not required.

Please, turn over!
4. Let $U$ be a bounded, open subset of $\mathbb{R}^2$ and let $L$ be the second-order linear partial differential operator defined by $Lu = -u_{x_1x_1} + \cos(x_1x_2)u_{x_1x_2} - u_{x_2x_2} + x_2u_{x_1} + x_1u_{x_2}$. Show that $L$ is uniformly elliptic and that $u \equiv 0$ is the only solution of class $C^2(U) \cap C(\overline{U})$ of the nonlinear boundary-value problem

$$
Lu = -u^3 \quad \text{in } U,
$$

$$
u = 0 \quad \text{on } \partial U.
$$

5. Consider the following boundary-value problem for Laplace’s equation in $\mathbb{R}^n$:

$$
\Delta u = 0 \quad \text{in } U^+,
$$

$$
u(x', 0) = g(x') \quad \text{for } |x'| < r,
$$

$$
u_{x_n}(x', 0) = 0 \quad \text{for } |x'| < r,
$$

where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $x = (x', x_n) \in \mathbb{R}^n$ and $U^+$ is the open half-ball $\{x \in \mathbb{R}^n : |x| < r, x_n > 0\}$. By the Cauchy-Kovaleskaya theorem this problem has a real analytic solution defined in the whole ball $B^0(0, r)$ for sufficiently small $r > 0$ if $g$ is real analytic at the origin. Show that it is a necessary condition that $g$ is real analytic in the set $|x'| < r$ for the above problem to have a solution $u \in C^1(\overline{U}^+) \cap C^2(U^+)$. If you don’t manage to show this, you can still get some points if you prove it under the stronger hypothesis $u \in C^2(\overline{U}^+)$. 

Hint: Recall that harmonic functions are real analytic in interior points.

Remark: This can be seen as a form of ill-posedness result.