Solutions

1. a) Separation of variables gives the solution

$$u(x,t) = \sin\left(\frac{x}{2}\right) e^{-\frac{1}{4}t}.$$ 

b) Suppose that $v$ is another solution and let $w = u - v$. Then $w$ solves the same problem as $u$ and $v$ but with the initial condition $w(x,0) = 0$. We have

$$\frac{d}{dt} \int_0^\pi w^2(x,t) \, dx = 2 \int_0^\pi w(x,t) w_t(x,t) \, dx = 2 \int_0^\pi w(x,t) w_{xx}(x,t) \, dx$$

$$= [2w(x,t)w_x(x,t)]_{x=0}^\pi - 2 \int_0^\pi (w_x(x,t))^2 \, dx = -2 \int_0^\pi (w_x(x,t))^2 \, dx$$

$$\leq 0$$

since $w$ vanishes at $x = 0$ and $w_x$ vanishes at $x = \pi$. Hence,

$$\int_0^\pi w^2(x,t) \, dx \leq \int_0^\pi w^2(x,0) \, dx = 0$$

for all $t \geq 0$. But this means that $w \equiv 0$ since $w$ is continuous.

2. This is a first-order equation, which can be solved using the method of characteristics. Since the equation is linear, the corresponding characteristic equations have the form

$$\begin{align*}
\dot{x} &= x, \\
\dot{y} &= 1, \\
\dot{z} &= 2yz.
\end{align*}$$

Solving first the equations for $x$ and $y$ we obtain $x(s) = ae^s$, $y(s) = s + b$, where $a$ and $b$ are parameters keeping track of which (projected) characteristic curve we are on. Although there are two parameters, we really only have a one-parameter family of geometrically distinct curves. The other parameter just keeps tells us where along the curve we are starting. We choose the parameter $b$ so that the solution passes through the $x$-axis when $s = 0$. Then $x(0) = a$ and $y(0) = 0$. The equation for $z$ now takes the form

$$\dot{z} = 2sz$$

with solution

$$z(s) = ce^{s^2}.$$ 

At $s = 0$ we have $z(0) = c$ and this should equal $g(a)$ by the initial condition. Thus

$$z(s) = g(a)e^{s^2}.$$
This is a solution formula in terms of the characteristic coordinates \(a\) and \(s\). In order to express the solution in terms of \(x\) and \(y\) we note that \(s = y\) and \(a = xe^{-x} = xe^{-x}\). Hence,

\[
u(x, y) = g(xe^{-x})e^{x^2}.
\]

We can verify that this is a solution by computing,

\[
u(x, 0) = g(x),
\]

\[
u_t(x, y) = e^{-y}g'(xe^{-x})e^{x^2}
\]

and

\[
u_t(x, y) = -xe^{-y}g'(xe^{-x})e^{x^2} + 2yg(xe^{-x})e^{x^2} = -xu_t(x, y) + 2yu(x, y).
\]

Since any point of the plane lies on a unique (projected) characteristic curve passing through the \(x\)-axis in a unique point, any solution must in fact be given by the above formula and therefore the solution is unique.

3. Since \(\phi\) is \(C^2\) with \(\phi(0) = 0\), it follows that for each \(R > 0\) there is a constant \(C \geq 0\) such that \(|\phi(u)| \leq C|u|\) whenever \(|u| \leq R\). Since \(u(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)\) it therefore follows that the integral \(\int_{\mathbb{R}^n} \phi(u(x, t)) \, dx\) is (absolutely) convergent for \(t \geq 0\). By a similar argument this integral is a continuously differentiable function of \(t\) and we can differentiate under the integral sign to obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \phi(u(x, t)) \, dx = \int_{\mathbb{R}^n} \phi'(u(x, t)) u_t(x, t) \, dx = \int_{\mathbb{R}^n} \phi'(u(x, t)) \Delta u(x, t) \, dx
\]

\[
= -\int_{\mathbb{R}^n} \phi''(u(x, t)) |\nabla u(x, t)|^2 \, dx,
\]

where there are no boundary terms since \(u(\cdot, t) \in \mathcal{S}(\mathbb{R}^n)\). Since \(\phi\) is \(C^2\) and convex, it follows that \(\phi''(u) \geq 0\) for all \(u\). Hence,

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \phi(u(x, t)) \, dx \leq 0,
\]

meaning that \(\int_{\mathbb{R}^n} \phi(u(x, t)) \, dx\) is decreasing. The \(L^p\) norms for \(p \geq 2\) are thus decreasing since \(\phi(u) = |u|^p\) satisfies the above assumptions.

4. In order to show that \(L\) is uniformly elliptic, we need to find a constant \(\theta > 0\) such that

\[
\xi_1^2 - \cos(x_1, x_2) \xi_1 \xi_2 + \xi_2^2 \geq \theta (\xi_1^2 + \xi_2^2)
\]

for each \((\xi_1, \xi_2) \in \mathbb{R}^2\) and \((x_1, x_2) \in U\). This follows (with \(\theta = 1/2\)) by noting that

\[
\xi_1^2 - \cos(x_1, x_2) \xi_1 \xi_2 + \xi_2^2 \geq \xi_1^2 - \xi_1 \xi_2 + \xi_2^2 \geq \frac{1}{2} (\xi_1^2 + \xi_2^2).
\]

Suppose now that \(u \in C^2(U) \cap C(\overline{U})\) is a solution to the problem which doesn’t vanish identically. Then \(u\) must either be strictly positive or strictly negative somewhere in \(U\). Suppose that it’s positive somewhere in \(U\) (the other case can be handled similarly). Then it must also have a strictly positive maximum attained at some point \(x^0 = (x_1^0, x_2^0) \in U\). We then have \(u_{x_1}(x^0) = u_{x_2}(x^0) = 0\) and \(u^3(x^0) > 0\). Hence,

\[
u_{x_1, x_1}(x^0) - \cos(x_1^0, x_2^0) u_{x_1, x_2}(x^0) + u_{x_2, x_2}(x^0) > 0.
\]
We define the function $v: B^0(0, r) \to \mathbb{R}$ by setting

$$v(x', x_n) = \begin{cases} u(x', x_n), & x_n \geq 0, \\ u(x', -x_n), & x_n < 0. \end{cases}$$

This is clearly a continuous function and we will show that it’s also harmonic in $B^0(0, r)$. To see this, let

$$w(x) = \frac{r^2 - |x|^2}{n \alpha(n) r} \int_{\partial B(0, r)} \frac{v(y)}{|x - y|^n} \, dS(y).$$

Then $w$ is harmonic (and therefore real analytic) in $B^0(0, r)$ with $w = v$ on $\partial B(0, r)$. In particular, $w = u$ on $\partial U_1^+ := \{x \in \mathbb{R}^n : |x| = r, x_n \geq 0\}$. Moreover,

$$w(x', x_n) = \frac{r^2 - |(x', -x_n)|^2}{n \alpha(n) r} \int_{\partial B(0, r)} \frac{v(y)}{|(x' - y', -x_n - y_n)|^n} \, dS(y)$$

$$= \frac{r^2 - |x'|^2}{n \alpha(n) r} \int_{\partial B(0, r)} \frac{v(y)}{|x - y|^n} \, dS(y)$$

$$= w(x).$$

where we have used the change of variables $y_n \mapsto -y_n$ and the fact that the sphere $\partial B(0, r)$ is symmetric with respect to the $x'$-plane. Hence, $w$ is even in $x_n$, and in particular

$$w_{x_n}(x', 0) = 0.$$

Thus $q = u - w$ solves the problem $\Delta q = 0$ in $U^+$ with $q = 0$ on $\partial U^+_1$ and $q_{x_n} = 0$ on $\partial U^+_2 := \{x \in \mathbb{R}^n : |x| < r, x_n = 0\}$. But this means that $q$ vanishes in $U^+$. Indeed, it cannot have an interior maximum or minimum unless it vanishes by the strong maximum principle. By Hopf’s lemma it also cannot have a maximum or minimum on $\partial U^+_2$ unless it vanishes since $q_{x_n} = 0$ on $\partial U^+_2$ (note that the edge $|x| = r, x_n = 0$ is not included and hence the interior ball condition is satisfied at every point of $\partial U^+_2$). Hence, the maximum and minimum must be attained on $\partial U^+_1$ where $q = 0$ and thus $u \equiv w$ in $U^+$. It therefore follows that $g(x') = w(x', 0)$ is real analytic in the set $|x'| < r$. Note also that $v \equiv w$ in $B(0, r)$ since both $w$ and $v$ are even in $x_n$, although we didn’t need to use this.

An alternative solution is to note that $v - w$ belongs to $H^1_0(B^0(0, r))$ and is a weak solution of Laplace’s equation in $B^0(0, r)$. It then follows that $v - w = 0$ almost everywhere, so that $v \equiv w$ (both functions are continuous).

If you instead assume that $u \in C^2(\overline{U^+})$ it suffices to show that the even extension $v$ is $C^2$ and harmonic in the whole ball. This can be done by computing the derivatives of $v$ for $x_n < 0$ and checking that they match the values from above on the $x'$-axis.