No aids allowed. Use the distributed paper sheets and write only on one side, with at most one problem per page. Fill in the cover sheet completely and initialize every sheet. Write legibly (in Swedish or English). Motivate your conclusions clearly and concisely; draw a picture if that helps to clarify your argument.

Test results will be posted on the course homepage before 17.00 on Friday August 23. For oral exams, contact the course leader.

1. Let

\[ \mathcal{E} = \left\{ \sum_{k=-N}^{N} c_k e^{ikx} : c_k \in \mathbb{C}; \, N \in \mathbb{Z}_+ \right\} \]

a) Show that \( \mathcal{E} \) is dense in \( C([-\pi, \pi], \mathbb{R}) \).

b) Is \( \mathcal{E} \) dense in \( C([-2\pi, 2\pi], \mathbb{R}) \)? Motivate your answer.

2. Let \( \mathcal{T} \) be a family of sets \( U \subseteq \mathbb{R}^4 \) such that

\[ U = \left\{ (x_1, x_2, x_3, x_4) : \, x_1 = r, \, (x_2, x_3, x_4) \in V \right\} \]

where \( r \in \mathbb{R} \) and \( V \) is open in the usual metric topology on \( \mathbb{R}^3 \).

a) Show that these sets form a base for a topology on \( \mathbb{R}^4 \).

b) Show that this topology larger than the standard topology on \( \mathbb{R}^4 \).

c) Which are the functions that are continuous with respect to this topology?

d) Show that this topology is not separable.

e) What are the relative topologies inherited from \( \mathcal{T} \) on the plane \( \{ x : \, x_1 = 0 \} \) and on the line \( L = \{ (x_1, 0, 0, 0) : \, x_1 \in \mathbb{R} \} \)?

3. Let \( X = \prod_{\alpha} X_{\alpha} \) be an infinite product with the usual product topology. Show that if \( \emptyset \neq A_\alpha \subseteq X_{\alpha} \) then the interior

\[ \text{int} \prod_{\alpha} A_{\alpha} \subseteq \prod_{\alpha} \text{int} A_{\alpha} \]

Show by a counterexample that one needs not have equality.

4. A mapping \( f : \, X \rightarrow X \) is idempotent if \( f \circ f = f \), i.e., \( f(f(x)) = f(x), \forall x \in X \).

Show that if \( f \) is continuous and idempotent on a Hausdorff space \( X \), then the image \( f(X) \) is closed. \textit{Hint:} show that if \( w \in f(X) \) then \( f(w) = w \). You get partial credits for showing this in the case of metric spaces.

5. Let \( D_0 = \{ (x, y) \in \mathbb{R}^2 : \, x^2 + y^2 < 1 \} \) be open unit disc, and let \( D_j \) be the union of \( D_0 \) and \( j \) disjoint points on the unit circle \( \partial D_0 \), all with the relative topology inherited from the usual metric topology of \( \mathbb{R}^2 \). Show that \( D_j \) is homeomorphic to \( D_k \) if and only if \( j = k \). \textit{Hint:} Start with \( j = 0 \) and \( k = 1 \), check the local properties of the spaces.