Introduction to Hardy spaces
CHAPTER 1

Preliminaries

1. Sequences and families of holomorphic functions

Given any sequence \((f_n)\) of holomorphic functions in the open set \(U \subset \mathbb{C}\) which converges uniformly on every compact subset of \(U\) to a function \(f : U \to \mathbb{C}\), it follows that \(f\) is also holomorphic on \(U\). We shall list some basic results about uniform convergence on compacts. Their proofs and additional information is deferred to Appendix A.

**Theorem 1.1.** Let \((f_n)\) be a sequence of holomorphic functions on the open connected set \(U \subset \mathbb{C}\) that converges uniformly on compacts to the (holomorphic) function \(f\). Let \(V\) be an open subset of \(U\) such that there exists \(n_0 \geq 1\) with \(f_n(z) \neq 0\) for all \(z \in V\) and \(n \geq n_0\). Then either \(f\) is the constant function 0, or \(f\) has no zeros in \(V\).

**Definition 1.1.** A family \(\mathcal{F}\) of holomorphic functions on the open set \(U \subset \mathbb{C}\) is called normal if every sequence \((f_n)\) in \(\mathcal{F}\) contains a subsequence \((f_{n_k})\) that converges uniformly on compact subsets to some (holomorphic) function \(f\) (not necessarily in \(\mathcal{F}\)!) 

**Theorem 1.2.** (Montel). A family \(\mathcal{F}\) of holomorphic functions on the open set \(U \subset \mathbb{C}\) is normal if it is uniformly bounded on compacts, that is, for every compact set \(K \subset U\) there exists a positive constant \(C_K\) such that for all \(f \in \mathcal{F}\) and all \(z \in K\) we have 

\[
|f(z)| \leq C_K.
\]

For uniform convergence on compacts we have the following useful criterion.

**Theorem 1.3.** (Vitali) Let \((f_n)\) be a sequence of holomorphic functions on the open connected set \(U \subset \mathbb{C}\). Suppose that \(\{f_n, \ n = 1, 2, \ldots\}\) is uniformly bounded on compacts and also that there is a nondiscrete subset \(A\) of \(U\) such that \((f_n)\) converges pointwise on \(A\). Then \((f_n)\) converges uniformly on compacts.

We want to apply this theorem to infinite products of bounded analytic functions.
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**Proposition 1.1.** Let \((f_n)\) be a sequence of holomorphic functions on the open set \(U \subset \mathbb{C}\) with the property that \(|f_n(z)| \leq 1\) for all \(z \in U\) and all \(n \in \mathbb{N}\). Let \((F_n)\) be defined by

\[
F_n(z) = \prod_{k=1}^{n} f_k(z), \quad z \in U
\]

and assume that there is \(z_0 \in U\) such that \((F_n(z_0))\) has a nonzero limit. Then \((F_n)\) converges uniformly on compacts to a nonzero holomorphic function \(F\) on \(U\) and each zero of \(F\) is the zero of some \(f_n\), \(n \in \mathbb{N}\).

**Proof.** Note that the family \(\{F_n, n \geq 1\}\) is uniformly bounded in \(U\) by 1, hence, it is a normal family by Montel’s theorem. As in the proof of Vitali’s theorem it suffices to show that if \((F_{n_k})\) and \((F'_{n_k})\) subsequences that converge uniformly on compacts to \(G\) and \(H\) respectively, then \(G = H\). Since \(|F_n| \geq |F_{n+1}|\), we can choose a subsequence \((n''_k)\) of \((n_k)\) such that \(n''_k \geq n'_k\) which implies

\[
|G(z)| = \lim_{k \to \infty} |F_{n''_k}(z)| \leq \lim_{k \to \infty} |F_{n'_k}(z)| = |H(z)|.
\]

Analogously, one shows that \(|H| \leq |G|\) and we obtain that \(|G| = |H|\). This implies that \(G = \alpha H\) for some \(\alpha \in \mathbb{C}\) with \(|\alpha| = 1\). But, by hypothesis, we have also \(G(z_0) = H(z_0) \neq 0\) which implies \(\alpha = 1\) and \(G = H\). To see the assertion about the zeros, note first that if \(K \subset U\) is compact then \(K \cap (\cup_n f_n^{-1}\{\{0\}\})\) is finite, otherwise \(F\) would be identically zero. Then if \(z\) is a zero of \(F\) that is not a zero of any \(f_n\) there is an open neighborhood of \(z\) that contains no zeros of \(f_n, n \geq 1\) and this leads to a contradiction by Theorem 1.1. \(\square\)

Let us denote from now on by \(\mathbb{D}\) the (open) unit disc (i.e. the disc centered at the origin and of radius one). Our next result will be frequently used in what follows and is a direct consequence of Proposition 1.4 combined with the following simple but important identity:

If \(z, w \in \mathbb{C}\) with \(\overline{w}z \neq 1\) then

\[
1 - \left| \frac{z - w}{1 - \overline{w}z} \right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2}.
\]

Indeed, this can be verified by a direct calculation (exercise!).

**Corollary 1.1.** Let \((a_n)\) be a sequence in \(\mathbb{D} \setminus \{0\}\). If \(\sum_n (1 - |a_n|) = \infty\) then the infinite product

\[
B(z) = \prod_{n=1}^{\infty} \frac{-\overline{a_n} z - a_n}{|a_n| 1 - \overline{a_n} z}
\]
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converges uniformly to zero on compact subsets of \( \mathbb{D} \). If \( \sum_n (1-|a_n|) < \infty \) then the product converges uniformly on compacts to a nonzero holomorphic function in \( \mathbb{D} \). Moreover, in this case, the zeros of \( B \) are precisely the points \( a_n, \ n \geq 1 \) and for fixed \( m \), the multiplicity of the zero \( a_m \) equals the number of occurrences of \( a_m \) in the sequence.

Proof. Assume that \( \sum_n (1-|a_n|) = \infty \) and let \( w \in \mathbb{D} \). Note that

\[
\prod_{n=1}^{N} \left| \frac{w-a_n}{1-\overline{a_n}w} \right|^2 = \exp \left( \sum_{n=1}^{N} \log \left| \frac{w-a_n}{1-\overline{a_n}w} \right|^2 \right) \geq \exp \left( \sum_{n=1}^{N} \left( \frac{|w-a_n|^2}{|1-\overline{a_n}w|^2} - 1 \right) \right).
\]

By (1.2) we have

\[
1 - \left| \frac{w-a_n}{1-\overline{a_n}w} \right|^2 = \frac{(1-|w|^2)(1-|a_n|^2)}{|1-\overline{a_n}w|^2} \geq \frac{(1-|w|^2)(1-|a_n|^2)}{(1+|w|)^2}
\]

which implies that

\[
\sum_{n=1}^{N} \left( \frac{|w-a_n|^2}{|1-\overline{a_n}w|^2} - 1 \right) \leq - \frac{1-|w|^2}{(1+|w|)^2} \sum_{n=1}^{N} (1-|a_n|^2) \to -\infty
\]

when \( N \to \infty \), and hence, the product converges to zero uniformly on compacts.

Now assume that \( \sum_n (1-|a_n|) < \infty \). From (1.2) we deduce that the factors \( f_n(z) = \frac{z-a_n}{|a_n|} \frac{z-a_n}{1-\overline{a_n}z} \) satisfy

\[
1 - |f_n(z)|^2 = \frac{(1-|z|^2)(1-|a_n|^2)}{|1-\overline{a_n}z|^2} > 0
\]

for all \( z \in \mathbb{D} \), that is \( |f_n(z)| < 1, \ z \in \mathbb{D} \). If the above sum converges, it is a standard matter to conclude that \( \lim_{n \to \infty} \prod_{k=1}^{n} |a_k| \) exists and is nonzero. Thus, the two assumptions in Proposition 1.1 are satisfied with \( z_0 = 0 \) and the convergence of the product follows. The statement about the zeros of \( B \) is also an immediate consequence of Proposition 1.1.

The functions considered in Corollary 1.1 are called Blaschke products and they play an important role in the theory of analytic functions in the unit disc. If we want to allow Blaschke products to have zeros at the origin as well (and we should) then the general definition reads as follows: A Blaschke product is a holomorphic function in \( \mathbb{D} \) of the form

\[
B(z) = z^m \prod_{n=1}^{\infty} \frac{-a_n}{|a_n|} \frac{z-a_n}{1-\overline{a_n}z},
\]
where \( m \) is a nonnegative integer, \((a_n)\) is a sequence in \( \mathbb{D} \) with \( \sum_n (1 - |a_n|) < \infty \).

**Exercise 1.** Show that the following families of holomorphic functions are normal.

(i) \( \mathcal{F}_1 = \{f_n, \ f_n(z) = z^n, \ |z| < 1, \ n \in \mathbb{N}\} \).

(ii) The family \( \mathcal{F}_2 \) of all power series \( f(z) = \sum_{n \geq 0} a_n z^n \) with \( |a_n| \leq 4, \ n \geq 0 \).

(iii) The family \( \mathcal{F}_3 \) of all holomorphic functions \( f \) in the open set \( U \subset \mathbb{C} \) with the property that

\[
\int_U |f(x + iy)| \, dx \, dy < 5.
\]

**Exercise 2.** Show that if \( \mathcal{F} \) is a normal family of analytic functions in the open set \( U \) then the families \( \mathcal{F}^{(n)} = \{f^{(n)}, \ f \in \mathcal{F}\} \) are also normal for all \( n \in \mathbb{N} \).

**Exercise 3.** Prove the converse of Montel’s theorem.

**Exercise 4.** Let \( f \) be analytic in \( \mathbb{D} \) and continuous in \( \overline{\mathbb{D}} \) such that \( |f(z)| \leq 147 \) for all \( z \in \mathbb{D} \). Show that if \( a_1, a_2, \ldots, a_n \) are zeros of \( f \) and \( B \) is the (finite) Blaschke product with zeros \( a_k, \ 1 \leq k \leq n \), then \( |f(z)| \leq 147|B(z)| \) for all \( z \in \mathbb{D} \).

### 2. Boundary behavior of power series

This section is a primary on the subject. It is intended to provide some intuition and motivation for the results that follow. For this reason, it will contain a number (four) unproved results that are important for our discussion.

Given a power series with a positive radius of convergence (in other words, an analytic function in a disc) we would like to investigate its behavior near the boundary points of the disc of convergence. As it is well known from calculus in 2 variables, our observations may depend heavily on the way we approach such points and this does indeed happen as the following example shows.

**Example 2.1.** Let \( f(z) = \exp(-\frac{1+z}{1-z}), \ z \in \mathbb{D} \). Then:
(i) \( f \) is bounded in \( \mathbb{D} \) and
\[
\lim_{r \to 1^-} f(r) = 0,
\]

(ii) For every angle \( A \) with vertex at 1, symmetric w.r.t. the segment \([0,1)\) and with opening less than \( \pi \) we have
\[
\lim_{z \to 1 \atop z \in A \cap \mathbb{D}} f(z) = 0,
\]

(iii) For every \( w \in \overline{\mathbb{D}} \) there exists a sequence \((z_n)\) in \( \mathbb{D} \) with \( z_n \to 1 \) and
\[
\lim_{n \to \infty} f(z_n) = w.
\]

To prove these assertions, let \( g(z) = 1 + \frac{z}{1-z}, \quad z \in \mathbb{D} \). Then clearly, \( |g(z)| \to \infty \) if and only if \( z \to 1 \). Let us show first that \( g(\mathbb{D}) \) equals the right half plane. Indeed, if \( g(z) = w \) and \( |z| < 1 \) then
\[
\Re w = \Re \frac{1 + z}{1 - z} = \frac{\Re(1 + z)(1 - z)}{|1 - z|^2} = \frac{1 - |z|^2}{|1 - z|^2} > 0.
\]

Conversely, if \( \Re w > 0 \), a similar computation shows that \( |z| < 1 \).

(i) Clearly, \( |f| = \exp(-\Re g) \leq 1 \) if \( \Re g > 0 \). The second assertion is obvious since \( \Re g(r) = \frac{1 + r}{1 - r} \to \infty \) when \( r \to 1 \).

(ii) Every \( z = re^{it} \in A \cap \mathbb{D} \) can be written in the form \( z = 1 + \rho e^{i\theta} \), where \( \rho > 0 \) and \( |\theta - \pi| \leq \alpha < \pi/2 \). Then
\[
r = (1 + \rho^2 + 2\rho \cos \theta)^{1/2}, \quad e^{it} = \frac{1 + \rho e^{i\theta}}{(1 + \rho^2 + 2\rho \cos \theta)^{1/2}}
\]
and
\[
\frac{e^{it} - 1}{1 - r^2} = -\frac{1 + \rho e^{i\theta} - (1 + \rho^2 + 2\rho \cos \theta)^{1/2}}{(1 + \rho^2 + 2\rho \cos \theta)^{1/2} \rho(\rho + 2 \cos \theta)}.
\]

Now the inequality satisfied by the angle \( \theta \) implies that this quantity stays bounded when \( \rho \to 0 \), or equivalently, \( z \to 1, z \in A \cap \mathbb{D} \). This is further equivalent to the fact that
\[(2.1) \quad \limsup_{re^{it} \to 1 \atop re^{it} \in A \cap \mathbb{D}} \frac{|f|}{1 - r} < \infty.
\]

With this fact at hand, the proof of (ii) is immediate. Indeed, from above we have that
\[
\Re g(re^{it}) = \frac{1 - r^2}{|1 - re^{it}|^2} = \frac{1 - r^2}{1 + r^2 - 2r \cos t} = \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2(t/2)}
\]
and from (2.1) it follows that
\[ \liminf_{r \to 1, r \in A \cap D} \text{Reg}(re^{it}) = \infty. \]

(iii) Clearly, it suffices to prove the statement for \( w \in \mathbb{D} \). Write \( w = re^{i\theta} \) and let \( w_n = -\log r + i(\theta + 2n\pi) \). Then \( \text{Re}w_n > 0 \) and there exists \( z_n \in \mathbb{D} \) with \( g(z_n) = w_n \) and since \( w_n \to \infty \) when \( n \to \infty \), it follows that \( \lim_{n \to \infty} z_n = 1 \). Obviously, \( f(z_n) = w \) for all \( n \).

Motivated by the above example we introduce the following terminology.

**Definition 2.1.** Let \( f \) be a complex-valued function defined on \( \mathbb{D} \) and let \( \zeta \in \partial \mathbb{D} \).

(i) We say that \( f \) has the **radial limit** \( L \) at \( \zeta \) if
\[ \lim_{r \to 1} f(r\zeta) = L. \]

(ii) We say that a sequence \( (z_n) \) in \( \mathbb{D} \) converges **nontangentially** to a point \( \zeta \in \partial \mathbb{D} \) if there is an angle with vertex at \( \zeta \), symmetric w.r.t. the ray joining \( \zeta \) to the origin and with opening less than \( \pi \) such that \( z_n \) belongs to this angle for all \( n \) sufficiently large.

(iii) We say that the function \( f \) has the **nontangential limit** \( L \) at the point \( \zeta \in \partial \mathbb{D} \) if for every sequence \( (z_n) \) in \( \mathbb{D} \) that converges nontangentially to \( \zeta \) we have
\[ \lim_{n \to \infty} f(z_n) = L. \]

To avoid confusions about the sets involved in the definition of nontangential convergence one usually considers **Stolz angles** which are sets obtained in the following way: Take a disc of radius \( 0 < \sigma < 1 \) centered at the origin and denote by \( \Gamma_\sigma(\zeta) \) the union of all line segments that join a point in that disc and \( \zeta \). In other words, \( \Gamma_\sigma(\zeta) \) is the convex hull of the above disc and the point \( \zeta \). Then a sequence \( (z_n) \) in \( \mathbb{D} \) converges nontangentially to a point \( \zeta \in \partial \mathbb{D} \) if there exists \( 0 < \sigma < 1 \) such that \( z_n \in \Gamma_\sigma(\zeta) \) for all \( n \) sufficiently large.

**Remark 2.1.** Recall that in general, \( 1 - |z| \leq |z - \zeta| \) if \( z \in \mathbb{D} \) and \( \zeta \in \partial \mathbb{D} \) and of course, if \( z \to \zeta \) it may happen that \( 1 - |z| \) goes much faster to 0 than \( |z - \zeta| \). The point is that if \( z \) approaches \( \zeta \) nontangentially this cannot happen, i.e. the quotient \( (1 - |z|)/|z - \zeta| \) stays bounded below. This statement is equivalent to the one in the next exercise and a proof was already pointed out in the previous example.
**Exercise 1.** Prove in detail that a sequence \((z_n)\) in \(\mathbb{D}\) with \(z_n = r_ne^{i\theta_n}\), converges nontangentially to a point \(\zeta = e^{i\theta} \in \partial \mathbb{D}\) if and only if \(\frac{|\theta_n - \theta|}{1-r_n}\) stays bounded when \(n \to \infty\).

**Exercise 2.** Show that the power series
\[
f(z) = \sum_{n=0}^{\infty} z^n, \quad z \in \mathbb{D}
\]
has no radial limit at any point \(e^{it}\) with \(t/\pi \in \mathbb{Q}\).

Throughout in what follows \(m\) will denote the normalized arclength-measure on the unit circle \(T = \partial \mathbb{D}\), which can be identified with the normalized Lebesgue measure on \([0, 2\pi]\).

It turns out that the radial limits of a power series provide much less information about the function inside the disc than the nontangential limits. Thus, for example, it is known that there exist analytic functions \(f\) in the unit disc that are not identically zero and yet satisfy
\[
\lim_{r \to 1} f(re^{it}) = 0 \quad \text{a.e. on } [0, 2\pi]
\]
(see [Col]). An even more dramatic example has been constructed by Kahane and Katznelson [KK]. They show that there exist such ”bad” functions \(f\) that satisfy in addition a growth restriction of the form
\[
|f(z)| \leq C(1-|z|)^{-\alpha}
\]
for \(\alpha > 0\) fixed but arbitrary!

On the other hand, this cannot happen with nontangential limits. This is essentially the content of a famous theorem of Privalov that is stated below without proof (for a proof see [Koo] ).

**Theorem 2.1. (Privalov)** If \(f\) is analytic in \(\mathbb{D}\) and has the nontangential limit zero on a set of positive measure on the unit circle \(T\), then \(f\) vanishes identically.

**Exercise 3.** Given an analytic function \(f\) in \(\mathbb{D}\) show that the set of points \(t \in [0, 2\pi]\) with the property that \(f\) has a nontangential limit at \(e^{it}\) is measurable.

**Example 2.2.** If \(\mu > 1\) is fixed then the infinite product
\[
f(z) = \prod_{n=0}^{\infty} (1 + \mu z^n)
\]
converges uniformly on compacts on $\mathbb{D}$ to a nonzero holomorphic function with the property that the set of points $t \in [0, 2\pi]$ such that $f$ has a nontangential limit at $e^{it}$ has measure zero.

To prove the convergence note first that every compact subset of $\mathbb{D}$ is contained in some disc centered at the origin with radius $r < 1$. Now for such $r$ and $|z| < r$ we have $|1 + \mu z^n| < 1 + \mu r^n \leq \exp(\mu r^n)$, which implies that the functions

$$f_N(z) = \prod_{n=0}^{N} (1 + \mu z^n)$$

satisfy

$$|f_N(z)| < \exp\left(\sum_{n=0}^{\infty} \mu r^n\right) = \exp(\frac{\mu}{1 - r}).$$

Thus, the family $\{f_N, \ N \geq 1\}$ is uniformly bounded on compacts and hence normal, by Montel’s theorem. Moreover, exactly the same argument as above shows that $(f_N(r))$ converges for every $0 < r < 1$ and by Vitali’s theorem the convergence assertion follows. Finally, using the inequality $1 + x \geq e^{-x}$ we see that for $0 < r < 1$

$$f(r) \geq \exp\left(-\sum_{n=0}^{\infty} \mu r^n\right) = \exp(-\frac{\mu}{1 - r}),$$

i.e. $f$ is not identically zero.

On the other hand, every point

$$z_{nk} = \mu^{-1/n} e^{\frac{(2k+1)i}{n}}, \quad n, k \in \mathbb{N}$$

is a zero of the function $f$ because $z_{nk}^n = -1/\mu$. We claim that for every $t \in [0, 2\pi]$ there exists a subsequence of $(z_{nk})_{n,k \geq 1}$ that converges nontangentially to $e^{it}$. Indeed, given $n \geq 1$ we can find $0 \leq k_n < n$ such that $|t - \frac{2k_n+1}{n}\pi| < \frac{\pi}{n}$. Then clearly, $z_{nk_n} = \mu^{-1/n} e^{\frac{(2k_n+1)i}{n}} \to e^{it}$ and since

$$\frac{|t - \frac{2k_n+1}{n}\pi|}{1 - \mu^{-1/n}} < \frac{\pi}{n(1 - \mu^{-1/n})} \to \frac{\pi}{\log \mu}$$

it follows by Exercise 1 that $(z_{nk_n})$ converges nontangentially to $e^{it}$. This shows that whenever $f$ has a nontangential limit at some point $e^{it}$ the limit must be zero. But by Privalov’s theorem the set of these points must have measure zero (recall that the set in question is measurable!).

**Example 2.3.** There exist analytic functions $f$ in $\mathbb{D}$ that have a radial limit but no nontangential limit at almost every boundary point!
Indeed, if we consider a nonconstant function $f$ that has radial limits zero almost everywhere on the boundary, the set of boundary points where $f$ has nontangential limits must have measure zero otherwise this would contradict Privalov’s theorem.

It will turn out that such an erratic behavior near the boundary points as described in the previous examples, becomes impossible if functions in question satisfy an appropriate growth restriction. This is an important point of view because, as we shall see in the sequel, these growth restrictions are very easy to define and provide large classes (spaces) of analytic functions that behave nicely near the boundary.

Theorem 2.2. Let $f$ be analytic and bounded in $\mathbb{D}$. If $f$ has a radial limit at a boundary point $e^{it}$ then $f$ has a nontangential limit at that point and these limits coincide.

Proof. By a translation and a rotation it will be sufficient to prove the following statement. For $a > 0, 0 < b < \pi/2$ let $\Gamma_{a,b}$ be the set of points $z = re^{is} \in \mathbb{C}$ with $0 < r < a$ and $|s| < b$. If $f$ is analytic and bounded in $\Gamma_{a,b}$ and $\lim_{r \to 0} f(r) = L$ then for $0 < c < b$

$$\lim_{z \to 0, z \in \Gamma_{0,c}} f(z) = L.$$ 

To this end, consider the sequence $(f_n)$ with $f_n(z) = f(z/n)$, $z \in \Gamma_{a,b}$ and apply Vitali’s theorem. The family $\{f_n, \ n \geq 1\}$ is uniformly bounded in $\Gamma_{a,b}$ and if $0 < r < a$

$$\lim_{n \to \infty} f_n(r) = \lim_{n \to \infty} f(\frac{r}{n}) = L.$$ 

Since the set $(0,a)$ is not discrete in $\Gamma_{a,b}$ Vitali’s theorem implies that $(f_n)$ converges uniformly on compacts to the constant function $L$. Now let $(z_k)$ be a sequence in $\Gamma_{a,c}$, $0 < c < b$ that converges to 0. We want to show that $f(z_k) \to L$. Choose a sequence $(n_k)$ of integers such that $n_k|z_k| \in (a/2, a)$ for sufficiently large $k$ (for example, the choice $n_k + 1 = \text{integer part of } \frac{a}{|z_k|}$ will do). Then the points $w_k = n_kz_k$ satisfy $a/2 < |w_k| < a$, $|\arg w_k| < c$ for sufficiently large $k$ which shows that these points $w_k$ lie in a compact subset of $\Gamma_{a,b}$. Then by the above reasoning we have

$$L = \lim_{k \to \infty} f_{n_k}(w_k) = \lim_{k \to \infty} f(\frac{w_k}{n_k}) = \lim_{k \to \infty} f(z_k)$$

and the result follows. \qed
Exercise 3. Prove the following statement which provides an improvement of Theorem 2.2: Let $\gamma$ be an arc (continuous image of an interval) with one endpoint $e^it$ and all other points contained in some Stolz angle $\Gamma_\sigma(e^it)$. If $f$ is bounded and analytic in $\mathbb{D}$ and $f(z)$ has the limit $L$ as $z$ approaches $e^it$ along $\gamma$, then $f$ has the nontangential limit $L$ at $e^it$.

The natural and basic question that arises is when do nontangential limits of an analytic function exist? The following result is a classical sufficient condition for the existence of such a limit due to Abel.

**Theorem 2.3.** (Abel) Let $f(z) = \sum_{n \geq 0} a_n z^n$ be a convergent power series in $\mathbb{D}$. Assume that $\zeta \in \partial \mathbb{D}$ is such that the series $\sum_{n \geq 0} a_n \zeta^n$ converges and let $L$ be the value of the sum. Then $f$ has the nontangential limit $L$ at $\zeta$.

**Proof.** We may assume without loss of generality that $\zeta = 1$. Let $S_n = \sum_{k=n}^{\infty} a_n$ and note that by hypothesis $S_0 = L$ and $S_n \to 0$, $n \to \infty$. In particular, $(S_n)$ is bounded, say $|S_n| \leq M$ for all $n$. Now write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (S_n - S_{n+1})z^n.$$ 

Since $|S_n| \leq M$ it follows that $\sum_{n=0}^{\infty} S_n z^n$ converges for all $z \in \mathbb{D}$ and thus,

$$f(z) = \sum_{n=0}^{\infty} (S_n - S_{n+1})z^n = \sum_{n=0}^{\infty} S_n z^n - \sum_{n=0}^{\infty} S_{n+1} z^n = S_0 + \sum_{n=1}^{\infty} S_n (z^n - z^{n+1}).$$

This trick is frequently called summation by parts. Let $\varepsilon > 0$ and $n_0$ be such that $|S_n| < \varepsilon$ for $n \geq n_0$. Recall also that $S_0 = L$. Then from the last equality we get

$$|f(z) - L| \leq |z-1| \sum_{n=1}^{\infty} |S_n||z^{n-1}| \leq M|z-1| \sum_{n=1}^{n_0} |z|^{n-1} + \varepsilon|z-1| \sum_{n=n_0+1}^{\infty} |z|^{n-1} \leq Mn_0|z-1| + \varepsilon|z-1|(1 - |z|)^{-1}.$$ 

Now let $(z_k)$ be a sequence in $\mathbb{D}$ that converges nontangentially to 1. Recall that in this case there is a constant $K > 0$ such that $|z_k - 1|(1 - |z_k|)^{-1} < K$ and use the last estimate to obtain

$$\limsup_{k \to \infty} |f(z_k) - L| \leq \varepsilon K,$$

for every $\varepsilon > 0$, which finishes the proof. \qed
Of course, at the first sight the condition in Abel’s theorem seems difficult to test and has little to do with a growth restriction. Nevertheless there is a deep result in analysis that yields a global existence theorem for nontangential limits of power series with square summable coefficients. The following famous theorem was proved by L. Carleson.

**Theorem 2.4.** (Carleson) Let \( f(z) = \sum_{n \geq 0} a_n z^n \) be a power series with square summable coefficients, that is,

\[
\sum_{n \geq 0} |a_n|^2 < \infty.
\]

Then the series \( \sum_{n \geq 0} a_n \zeta^n \) converges for almost every \( \zeta \in \partial \mathbb{D} \).

It is a simple exercise to show that such power series converge in the unit disc. Thus, by Abel’s theorem every power series with square summable coefficients has a nontangential limit almost everywhere on the unit circle. Let us now show that the condition on the coefficients really is a growth restriction. This is a consequence of Parseval’s formula.

**Theorem 2.5.** Let \( f(z) = \sum_{n \geq 0} a_n z^n \) be a convergent power series in \( \mathbb{D} \). Then

\[
\sum_{n \geq 0} |a_n|^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = \lim_{r \to 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi}.
\]

**Proof.** Write

\[
|f(re^{it})|^2 = \left( \sum_{n=0}^{\infty} a_n r^n e^{int} \right) \left( \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int} \right) = \sum_{m,n=0}^{\infty} a_n \overline{a_m} r^{m+n} e^{i(m-n)t}
\]

and note that the last sum converges uniformly on \([0, 2\pi]\). But then we can interchange limit and integration to obtain

\[
\int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} = \sum_{m,n=0}^{\infty} a_n \overline{a_m} r^{m+n} \int_0^{2\pi} e^{i(m-n)t} \frac{dt}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},
\]

because \( \int_0^{2\pi} e^{ikt} \frac{dt}{2\pi} = \delta_{k0} \), that is, it equals 0 if \( k \neq 0 \) and 1 if \( k = 0 \). The rest of the proof is a routine exercise based on the fact that the integrals involved here increase with \( r \) as the last identity shows. \( \square \)

Thus, we can reformulate the result on nontangential limits in terms of the growth restriction we just found.

**Corollary 2.1.** Let \( f \) be analytic in \( \mathbb{D} \) and suppose that

\[
\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < \infty.
\]
Then \( f \) has nontangential limits a.e. on the unit circle.

The above result is classical, and, as we shall see in the sequel, its proof does not need the "heavy artillery" provided by Carleson’s theorem. We close this section with a result which shows that, in terms of Taylor coefficients, the square summability condition in Corollary 2.6 cannot be improved. The theorem below is a special case of a much stronger result of Khinchin and Kolmogorov (see [Du], p.).

**Theorem 2.6.** Let \( f(z) = \sum_{n \geq 0} a_n z^n \) be a convergent power series in \( \mathbb{D} \) such that
\[
\sum_{n \geq 0} |a_n|^2 = \infty.
\]

Then there exists a sequence \((\varepsilon_n)\) with \( \varepsilon_n = \pm 1, \ n = 0, 1, 2 \ldots \) such that the power series
\[
g(z) = \sum_{n \geq 0} \varepsilon_n a_n z^n
\]
has no radial limit almost everywhere on the unit circle.
CHAPTER 2

Poisson integrals

1. Harmonic functions

Recall that a twice differentiable complex-valued function \( u \) defined on some open set \( G \subset \mathbb{C} \) is called harmonic if it satisfies the Laplace equation

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } G.
\]

Of course, holomorphic functions are harmonic. Moreover, a very important class of examples of real-valued harmonic functions are the real (and imaginary) parts of holomorphic functions in \( G \). This follows from a very simple computation with the Cauchy-Riemann equations. Conversely, if \( G \) is simply connected (roughly speaking, this means it has no holes), it turns out that every real-valued harmonic function in \( G \) is the real part of a holomorphic function in \( G \). This is a classical result that needs to be proved. In order to avoid technicalities regarding the definition of a simply connected domain we will use an equivalent property.

1.1. Theorem Suppose that \( G \) is an open connected set in \( \mathbb{C} \) with the property that every holomorphic function on \( G \) has a primitive in \( G \). Then every real-valued harmonic function in \( G \) is the real part of a holomorphic function in this open set.

Proof. Observe first that if \( u \) is harmonic on \( G \) then the function \( h = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \) satisfies the Cauchy-Riemann equations in \( G \). Indeed, since \( u \) is harmonic we have

\[
\frac{\partial}{\partial x} \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( - \frac{\partial u}{\partial y} \right),
\]

and

\[
\frac{\partial}{\partial y} \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial xy} = - \frac{\partial}{\partial x} \left( - \frac{\partial u}{\partial y} \right).
\]
Let $f$ be a primitive of the holomorphic function $h$, i.e. $f' = h$. Then
\[
\frac{\partial}{\partial x} \text{Re} f = \text{Re} f' = \frac{\partial u}{\partial x}
\]
and
\[
\frac{\partial}{\partial y} \text{Re} f = -\text{Im} f' = \frac{\partial u}{\partial y}.
\]
Thus, $\text{Re} f - u$ is constant in $G$ and the result follows.

The simplest example that comes to mind of a domain $G$ as above, is a disc.

The theorem has the following reformulation: If $G$ is as above and $u$ is harmonic in $G$ then there exists a harmonic function $v$ in $G$ such that $u + iv$ is analytic in $G$.

**Exercise 1.** Show that $v$ is harmonic and it is uniquely determined up to an additive constant (i.e. if $v_1, v_2$ have this property then $v_1 - v_2$ is constant in $G$).

Such a function $v$ is called a *harmonic conjugate* of $u$, while the association $u \mapsto f = u + iv$ is sometimes called *analytic completion*.

**Exercise 2** Given function of the form
\[
u(z) = \sum_{k,n=0}^{N} a_{kn} z^k \overline{z}^n, \quad z \in \mathbb{C}
\]
show that $u$ is harmonic if and only if $a_{kn} = 0$ whenever $k$ and $n$ are both nonzero.

### 2. Representation by Poisson integrals

Since harmonic functions are so close to the analytic ones it is natural to try to find an analogue of Cauchy’s formula for these functions. Here is a first attempt for harmonic functions in the unit disc.

If $u$ is harmonic and real-valued in a disc containing $\mathbb{D}$ and $f$ is an analytic completion of $u$ then by Cauchy’s formula we have for all
2. REPRESENTATION BY POISSON INTEGRALS

\( z \in \mathbb{D} \)

\[ u(z) = \text{Re} f(z) = \text{Re} \left( \frac{1}{2\pi i} \int_{|\zeta| = 1} f(\zeta) d\zeta \right) \]

\[ = \text{Re} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \frac{f(e^{it})}{e^{it} - z} dt \right). \]  

(2.1)

This is an integral representation of \( u \) which, unfortunately, involves one of its harmonic conjugates and therefore needs to be improved.

**Lemma 2.1.** If \( u \) is harmonic in a disc containing \( \mathbb{D} \) then for all \( z \in \mathbb{D} \) we have

\[ u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(e^{it}) dt. \]

**Proof.** The trick is to insert in the last integral in (2.1) the harmless term \( z f(e^{it})/(e^{-it} - \overline{z}) \), where \( f \) is again an analytic function whose real part equals \( u \). Note that for every integer \( n \geq 0 \) we have

\[ \int_{0}^{2\pi} \frac{e^{int} dt}{e^{-it} - \overline{z}} = \int_{0}^{2\pi} \frac{e^{i(n+1)t} dt}{1 - e^{it}\overline{z}} = -i \int_{|\zeta| = 1} \frac{\zeta^n d\zeta}{1 - \zeta \overline{z}} = 0, \]

by Cauchy’s Theorem. Thus, if we write

\[ f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int} \]

then from the fact that the power series converges uniformly on the unit circle we obtain

\[ \int_{0}^{2\pi} \frac{f(e^{it}) \overline{z} dt}{e^{-it} - \overline{z}} = \sum_{n=0}^{\infty} a_n \int_{0}^{2\pi} \frac{e^{int} \overline{z} dt}{e^{-it} - \overline{z}} = 0, \]

which shows that the term inserted in the last integral in (2.1) is indeed harmless. Therefore, from (2.1) we obtain

\[ u(z) = \text{Re} \left( \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{it}) \left( \frac{e^{it}}{e^{it} - z} + \frac{\overline{z}}{e^{-it} - \overline{z}} \right) dt \right). \]

Now note also that

\[ \frac{e^{it}}{e^{it} - z} + \frac{\overline{z}}{e^{-it} - \overline{z}} = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad z \neq e^{it}, \]

and that the right hand side of this equality is always real (it becomes also positive if \( |z| < 1 \)). Then

\[ u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \text{Re} f(e^{it}) dt \]
and the result follows. □

Observe that this lemma gives a much better integral representation of a harmonic function. The only problem that remains is to weaken the very restrictive hypothesis that \( u \) is actually harmonic in a larger disc. A first step in this direction is an immediate application of the theorem. is given in the next theorem.

**Corollary 2.1.** Let \( u \) be harmonic in \( \mathbb{D} \) and continuous in the closed disc \( \overline{\mathbb{D}} \). Then

\[
 u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(e^{it}) dt.
\]

**Proof.** For \( 0 < r < 1 \) fixed but arbitrary, consider the dilation \( u_r \) of \( u \) defined in \( \mathbb{D} \) by \( u_r(z) = u(rz) \). Clearly, \( u_r \) is harmonic in the disc of radius \( 1/r \) centered at the origin, so that Lemma 2.1 gives for every \( z \in \mathbb{D} \)

\[
 u_r(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u_r(e^{it}) dt.
\]

Now let \( r \to 1 \) and note that \( u_r(z) \to u(z) \), \( z \in \mathbb{D} \) and also, that the functions \( t \mapsto u_r(e^{it}), \ t \in [0, 2\pi] \) converge uniformly to \( t \mapsto u(e^{it}), \ t \in [0, 2\pi] \), by the continuity assumption. This implies that for fixed \( z \in \mathbb{D} \)

\[
 \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u_r(e^{it}) dt \to \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} u(e^{it}) dt, \quad r \to 1
\]

and we are done. □

The continuity assumption on \( u \) can be relaxed using some results from integration theory and functional analysis.

**Theorem 2.1.** Let \( u \) be harmonic in \( \mathbb{D} \) and suppose that there exist constants \( C > 0 \) and \( 1 \leq p \leq \infty \) such that for all \( r \in (0, 1) \) we have

\[
 \int_0^{2\pi} |u(re^{it})|^p \frac{dt}{2\pi} \leq C,
\]

if \( p < \infty \), or

\[
 \sup_{t \in [0, 2\pi]} |u(re^{it})| \leq C,
\]

if \( p = \infty \). Then:

(i) If \( p > 1 \), there exists a unique \( \tilde{u} \in L^p(\mathbb{D}) \) such that

\[
 u(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \tilde{u}(e^{it}) \frac{dt}{2\pi}.
\]

(ii) If \( p = 1 \), there exists a unique \( \tilde{u} \in L^1(\mathbb{D}) \) such that

\[
 u(z) = \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \tilde{u}(e^{it}) \frac{dt}{2\pi}.
\]
(ii) If $p = 1$, there exists a unique finite Borel measure $\mu$ on $\mathbb{T}$ such that

$$ u(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta). $$

**Proof.** We shall prove first the existence part. The argument goes exactly as the one in Corollary 2.1. More precisely, we consider the dilations $u_r, 0 < r < 1$ of $u$, apply Lemma 2.1 and let $r \to 1$.

(i) Assume first that $p < \infty$, and use the assumption in the statement together with the fact that $L^p(m)$ is reflexive, to find a sequence $(r_n)$ tending to 1 such that $(u_{r_n})$ converges weakly in $L^p(m)$ to $\tilde{u} \in L^p(m)$. Since the functions

$$ P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} $$

are bounded in $t$, we obtain the representation in the statement. When $p = \infty$ we only use weak-star sequential compactness instead of reflexivity to arrive at the same result. (ii) Here we use the fact that for fixed $z \in \mathbb{D}$ the function $P_z(\cdot)$ is continuous on $\mathbb{T}$ and since the measures $u_r dm$ have bounded total variation, we can find a sequence that converges weak-star in the dual of $C(\mathbb{T})$ to some finite Borel measure $\mu$ on $\mathbb{T}$.

It remains to show that $\tilde{u}, \mu$ are unique. A direct computation shows that

$$ P_z(e^{it}) = 1 + 2\text{Re} \frac{ze^{-it}}{1-ze^{-it}} = 1 + 2\sum_{n=0}^{\infty} z^n e^{-int}, $$

where the series converges uniformly on $[0, 2\pi]$. Since the equality in the statement holds for all $z \in \mathbb{D}$ we see that the Fourier coefficients of $\tilde{u}, \mu$ are uniquely determined, i.e. $\tilde{u}, \mu$ are unique. \qed

The kernel

$$ P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} $$

is called the Poisson kernel, and a harmonic function of the form described in the above theorem is called the Poisson integral of $\tilde{u}$, or $\mu$.

The simplest example that (i) cannot hold for $p = 1$ are the Poisson kernels themselves. For example, $P_z(1)$ satisfies the theorem with $p = 1$ and is the Poisson integral of the Dirac measure at 1 which is uniquely determined by this assertion.

**Corollary 2.2.** Let $u$ be harmonic and nonnegative in $\mathbb{D}$. Then there exists a unique nonnegative Borel measure $\mu$ on $\mathbb{T}$ such that $\mu(\mathbb{T}) =$
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\( u(0) \) and

\[
  u(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta).
\]

**Proof.** A nonnegative harmonic function \( u \) in \( \mathbb{D} \) satisfies the assumption in part (ii) of Theorem 2.1 since

\[
  \int_{0}^{2\pi} |u(re^{it})| \frac{dt}{2\pi} = \int_{0}^{2\pi} u(re^{it}) \frac{dt}{2\pi} = u(0).
\]

Since the measure \( \mu \) given by the theorem is the weak-star limit of nonnegative measures, it will be nonnegative as well. \( \square \)

**Corollary 2.3.** Let \( f \) be analytic with nonnegative real part in \( \mathbb{D} \). Then there exists a unique nonnegative Borel measure \( \mu \) on \( \mathbb{T} \) such that \( \mu(\mathbb{T}) = \text{Re} f(0) \) and

\[
  f(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).
\]

The above result is called the Herglotz representation of functions with positive real part.

**Exercise 1** Prove Corollary 2.3.

**Exercise 2** Prove that a harmonic function in an open connected subset \( G \) of \( \mathbb{C} \) which has a local minimum, or a local maximum in \( G \) is constant.

3. The nontangential maximal function

Given a function \( f : \mathbb{D} \to \mathbb{C} \) and \( \sigma \in (0, 1) \), the nontangential maximal function of \( f \) is defined on \( \mathbb{T} \) by

\[
  f^*(\zeta) = \sup_{z \in \Gamma_{\sigma}(\zeta)} |f(z)|.
\]

The definition depends on \( \sigma \), but is customary to drop this parameter in the notation. In all assertions \( \sigma \in (0, 1) \) is fixed but arbitrary, and the constants appearing in estimates will depend on the choice of \( \sigma \). The aim of this section is to compare the nontangential maximal function of Poisson integrals of functions \( h \in L^1(m) \) with the well-known Hardy-Littlewood maximal function of \( h \), defined by

\[
  h^\dagger(e^{i\theta}) = \sup_{t \in [0, \pi]} \frac{1}{2t} \int_{\theta - t}^{\theta + t} |h(e^{i\sigma})| ds.
\]
The Hardy-Littlewood maximal function plays a crucial role in $L^p$-estimates as it is shown by the following classical theorem essentially due to Hardy and Littlewood, which we state without proof.

**Theorem 3.1.** For $p > 1$, $h \hat{} \in L^p(m)$, whenever $h \in L^p(m)$, and there exists $C_p > 0$ such that

$$\|h\hat{\|}_p \leq C_p\|h\|_p.$$  

If $h \in L^1(m)$ there exists $C_1 > 0$ such that for all $t > 0$ we have

$$m(\{\zeta \in \mathbb{T}: h\hat{}(\zeta) > t\}) < C_1 \frac{\|h\|_1}{t}.$$  

We begin with some simple properties of the Poisson kernel which are listed below.

**Proposition 3.1.** (i) $P_z(e^{it}) > 0$, for all $z \in \mathbb{D}$ and $t \in [0, 2\pi]$.

(ii) For all $t, \theta \in [0, 2\pi)$ with $t \neq \theta$

$$\lim_{z \to e^{i\theta}} P_z(e^{it}) = 0.$$  

In fact, if we regard $P_z(\cdot)$ as a family of functions on $\mathbb{T}$ then the above convergence is uniform on any set of the form $\{\zeta \in \mathbb{T} : |\zeta - e^{it}| > \delta\}$, where $\delta > 0$ is fixed.

(iii) For all $z \in \mathbb{D}, s \in [0, 2\pi], 0 < r < 1$, we have

$$\int_0^{2\pi} P_z(e^{it}) \frac{dt}{2\pi} = \int_0^{2\pi} P_{re^{i\theta}}(e^{is}) \frac{d\theta}{2\pi} = 1.$$  

(iv) If $z = re^{i\theta}$ then $P_z(e^{i(\theta + t)}) = P_z(e^{i(\theta - t)})$.

(v) If $\zeta \in \mathbb{T}$ and $z = re^{it} \in \Gamma_{\sigma}(\zeta)$, there exists $C > 0$ depending only on $\sigma$ such that for all $\lambda \in \mathbb{T}$ we have

$$P_z(\lambda) \leq CP_{r\zeta}(\lambda).$$  

(vi) If $r \in [0, 1), t \in [-\pi, 2\pi]$ then

$$t \frac{\partial}{\partial t} P_r(e^{it}) \leq 0,$$  

and consequently, there exists $C > 0$ such that

$$\int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} P_r(e^{it}) \right| \frac{dt}{2\pi} \leq C,$$  

for all $r \in [0, 1)$.
PROOF. (i), (ii), and (iv) are obvious. The first integral in (iii) equals 1 by an application of Lemma 2.1 $u \equiv 1$, the second equality follows from $P_{re^{i\phi}}(t) = P_{re^{i\theta}}$. To see (v), write
\[
P_z(\lambda) = \frac{1 - r^2}{|\lambda - z|^2} = \frac{1 - r^2}{|\lambda - r\zeta|^2} \frac{|\lambda - r\zeta|^2}{|\lambda - z|^2} = P_{r\zeta}(\lambda) \frac{|\lambda - r\zeta|^2}{|\lambda - z|^2}.
\]
Moreover,
\[
\frac{|\lambda - r\zeta|}{|\lambda - z|} \leq \frac{|\lambda - z| + |z - z| + r|z - \zeta|}{\lambda - z} \leq 1 + r + r \frac{|z - \zeta|}{1 - r},
\]
and the right hand side is bounded within $\Gamma_\sigma(\zeta)$ by a constant that depends only on $\sigma$. Finally, the first part of (vi) is immediate since
\[
t \frac{\partial}{\partial t} P_t(e^{it}) = -\frac{2r(1 - r^2) t \sin t}{|e^{it} - r|^2},
\]
while the second part follows from this together with integration by parts:
\[
\int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} P_t(e^{it}) \right| \frac{dt}{2\pi} = -\int_{-\pi}^{\pi} t \frac{\partial}{\partial t} P_t(e^{it}) \frac{dt}{2\pi} = -P_r(-1) + \int_{-\pi}^{\pi} P_t(e^{it}) \frac{dt}{2\pi} = \frac{-1 - r}{1 + r} + 1.
\]
□

We can now turn to the main result of the section.

THEOREM 3.2. There exists $C > 0$ such that, given $h \in L^1(\mathbb{T})$ with Poisson integral $u$, we have
\[
u^*(\zeta) \leq Ch^1(\zeta), \quad \zeta \in \mathbb{T}.
\]
In particular, if $h \in L^p(m)$ for some $p \in (1, \infty)$ then $u^* \in L^p(m)$, and there exists $C_p > 0$, depending only on $p$, such that
\[
\|u^*\|_{L^p(m)} \leq C_p\|h\|_{L^p(m)}
\]

PROOF. Let $z \in \Gamma_\sigma(e^{it})$, with $|z| = r \in [0, 1)$, and use Proposition 3.1 (v) to obtain the estimate
\[
(3.5) \quad |u(z)| \leq \int_{\mathbb{T}} P_z(\lambda)|h(\lambda)|dm(\lambda) \leq C \int_{\mathbb{T}} P_{re^{i\phi}}(\lambda)|h(\lambda)|dm(\lambda).
\]
Using again this result we can write
\[
\int_{\mathbb{T}} P_{re^i\theta}(\lambda) |h(\lambda)| \, dm(\lambda) = \int_{-\pi}^{\pi} P_r(e^{it}) |h(e^{i(\theta+t)})| \frac{dt}{2\pi} = \int_{0}^{\pi} P_r(e^{it}) |h(e^{i(\theta+t)})| - |h(e^{i(\theta-t)})| \frac{dt}{2\pi}.
\]

Finally, integration by parts gives
\[
\int_{\mathbb{T}} P_{re^i\theta}(\lambda) |h(\lambda)| \, dm(\lambda) = P_r(-1) \int_{-\pi}^{\pi} |h(e^{i(\theta+t)})| \frac{dt}{2\pi} \left. \frac{\partial}{\partial t} P_r(e^{it}) \right|_{-t}^{t} |h((e^{i(\theta+s)})| \frac{ds}{2\pi}.
\]

Clearly,
\[
\int_{-\pi}^{\pi} |h((e^{i(\theta+t)})| \frac{dt}{2\pi} \leq h^\dagger(e^{i\theta}), \quad \int_{-t}^{t} |h((e^{i(\theta+s)})| \frac{ds}{2\pi} \leq \frac{t}{\pi} h^\dagger(e^{i\theta}).
\]

By Proposition 3.1 (vi) we have that
\[- \int_{0}^{\pi} \frac{\partial}{\partial t} P_r(e^{it}) \int_{-t}^{t} |h((e^{i(\theta+s)})| \frac{ds}{2\pi} \leq C'h^\dagger(e^{i\theta}),
\]

hence,
\[
\int_{\mathbb{T}} P_{re^i\theta}(\lambda) |h(\lambda)| \, dm(\lambda) \leq C''h^\dagger(e^{i\theta}),
\]

and the result follows by the estimate (3.5).

### 4. Boundary behavior of Poisson integrals

In this section we will use the maximal function estimate to derive a series of results about Poisson integrals of integrable functions.

**Theorem 4.1.** If \( u \) is the Poisson integral of \( h \in L^1(m) \), then \( u \) has the nontangential limit \( h(\zeta) \) at almost every \( \zeta \in \mathbb{T} \).

**Proof.** Let \( \varepsilon > 0 \), and let \( g \in C(\mathbb{T}) \) with \( \|h - g\|_1 < \varepsilon \). By the Hardy-Littlewood theorem it follows that
\[
m(\{\zeta \in \mathbb{T} : (h - g)^\dagger(\zeta) > \sqrt{\varepsilon}\}) < C_1\sqrt{\varepsilon}.
\]

Let \( v \) denote the Poisson integral of \( g \). By Theorem 3.2 we have that
\[
m(\{\zeta \in \mathbb{T} : (u - v)^\star(\zeta) > C\sqrt{\varepsilon}\}) < C_1\sqrt{\varepsilon},
\]

where \( C > 0 \) is the constant given in that theorem. For a fixed integer \( n > 0 \) consider the set \( A_n \subset \mathbb{T} \) consisting of those points \( \zeta \) for which the nontangential "limsup" of \( |u - h(\zeta)| \) at \( \zeta \) exceeds \( \frac{1}{n} \). Since \( v \) is
continuous in \( \overline{D} \) and equals \( g \) on \( T \), it follows that if \( \varepsilon \) is sufficiently small, \( A_n \) is contained in the set
\[
\{ \zeta \in T : (u - v)^*(\zeta) > C_1 \sqrt{\varepsilon} \} \cup \{ \zeta \in T : |h(\zeta) - g(\zeta)| > \sqrt{\varepsilon} \}.
\]
This gives that \( m(A_n) < C_1 \sqrt{\varepsilon} \), and since \( \varepsilon \) was arbitrary we conclude that \( m(A_n) = 0 \), which completes the proof. \( \square \)

The result can be extended to Poisson integrals of finite Borel measures, but this requires an additional step.

**Lemma 4.1.** If \( u \) is the Poisson integral of a finite Borel measure \( \mu \) on \( T \) which is singular w.r.t. \( m \), then \( u \) has nontangential limits zero \( m \)-a.e.

**Proof.** We may assume without loss of generality that the measure \( \mu \) is positive. The existence of the nontangential limits follows from Theorem 4.1. Indeed, if \( f \) is analytic in \( \mathbb{D} \) with \( \text{Re} f = u \), then \( e^{-f} \) is analytic in and bounded \( \mathbb{D} \), in particular it is a bounded harmonic function. Then it is the Poisson integral of a function in \( L^\infty(m) \), hence by the above result, it has nontangential limits a.e. on \( T \). This implies that \( u = -\log |f| \) has nontangential limits \( u(\zeta) \geq 0 \) for almost every \( \zeta \in T \). Now recall that our measure \( \mu \) is the weak-star limit of the measures \( u_r dm \), where \( u_r(z) = u(rz) \), i.e. for every continuous function \( h \) on \( T \) we have that
\[
\lim_{r \to 1^-} \int_T h u_r dm = \int_T h \mu.
\]
On the other hand, by Fatou’s lemma we obtain for every nonnegative continuous function \( h \) on \( T \),
\[
\int_T u(\zeta) h(\zeta) dm(\zeta) \leq \liminf_{r \to 1^-} \int_T h u_r dm = \int h \mu.
\]
Since the characteristic function of any compact subset of \( T \) can be approximated pointwise by a sequence of nonnegative continuous functions, another application of Fatou’s lemma leads to
\[
\int_A u dm \leq \mu(A),
\]
for all compact subsets of \( T \). From the fact that \( \mu \) is singular we deduce that \( u = 0 \), \( m \)-a.e. \( \square \)

If \( u \) is the Poisson integral of the finite Borel measure \( \mu \) on \( T \) we can write
\[
d\mu = h dm + d\nu,
\]
where $h = \frac{du}{dm} \in L^1(m)$ is the Radon-Nykodim derivative of $\mu$ w.r.t. $m$, and $\nu$ is singular w.r.t. $m$. By Theorem 4.1 and the above lemma we obtain:

**Corollary 4.1.** If $u$ is the Poisson integral of the finite Borel measure $\mu$ on $\mathbb{T}$ then $u$ has the nontangential limit $\frac{du}{dm}(\zeta)$ at almost every $\zeta \in \mathbb{T}$.

**5. $h^p$-spaces**

Given $0 < p < \infty$ we denote by $h^p$ the vector space of all harmonic functions $u$ in $\mathbb{D}$ with

$$
(5.6) \quad \|u\|_p^p = \sup_{0 < r < 1} \int_0^{2\pi} |u(re^{it})|^p \frac{dt}{2\pi} < \infty.
$$

$h^\infty$ will denote the space of bounded harmonic functions $u$ in $\mathbb{D}$ with

$$
\|u\|_\infty = \sup_{z \in \mathbb{D}} |u(z)|.
$$

We will be interested in the case $p \geq 1$, when (5.6) defines a norm on $h^p$.

If $p > 1$, we know from the previous section that every $u \in h^p$ is the Poisson integral of its boundary function defined a.e. on $\mathbb{T}$ by $u(\zeta) =$ nontangential limit of $u$ at $\zeta$. We shall keep this notation for the boundary values throughout in what follows. If $u \in h^1$ we only know that $u$ is a Poisson integral of a (unique) finite Borel measure on $\mathbb{T}$, which we denote by $\mu_u$. Its total variation is $\|\mu_u\| = |\mu_u|(\mathbb{T})$.

**Theorem 5.1.** (i) If $1 < p \leq \infty$ and $u \in h^p$ we have

$$
\|u\|_p^p = \|u\|_{L^p(m)}^p = \int_{\mathbb{T}} |u|^p dm,
$$

and if $p = 1$, then

$$
\|u\|_1 = \|\mu_u\|.
$$

For $1 \leq p \leq \infty$, $h^p$ is a Banach space with the norm (5.6).

(iii) If $1 < p < \infty$, and $u \in h^p$, $u_r(z) = u(rz)$, $0 \leq r < 1$, then

$$
\lim_{r \to 1^-} \|u_r - u\|_p = 0.
$$

**Proof.** (i) The case $p = \infty$ is immediate

$$
|u(z)| \leq \int_{\mathbb{T}} P_z(\zeta)|u(\zeta)dm(\zeta) \leq \|u\|_\infty \int_{\mathbb{T}} P_z(\zeta)dm(\zeta) = \|u\|_\infty.
$$
If $1 < p < \infty$, note that $u(\zeta) = \lim_{r \to 1^-} u(r \zeta)$, $m$-a.e. on $\mathbb{T}$, hence Fatou’s lemma gives
\[
\int_{\mathbb{T}} |u|^p dm \leq \|u\|_p^p.
\]
On the other hand, since $u$ is the Poisson integral of its boundary values, we can apply by Hölder’s inequality to obtain
\[
|u(z)|^p = \left| \int_{\mathbb{T}} P_z(\zeta) u(\zeta) dm(\zeta) \right|^p \\
\leq \left( \int_{\mathbb{T}} P_z(\zeta) |u(\zeta)|^p dm(\zeta) \right)^p \cdot \left( \int_{\mathbb{T}} P_z(\zeta) dm(\zeta) \right)^{p-1} \\
= \int_{\mathbb{T}} P_z(\zeta) |u(\zeta)|^p dm(\zeta),
\]
so that
\[
\int_0^{2\pi} |u(re^{it})|^p \frac{dt}{2\pi} \leq \int_0^{2\pi} \int_{\mathbb{T}} P_z(\zeta) |u(\zeta)|^p dm(\zeta) \frac{dt}{2\pi} = \int_{\mathbb{T}} |u|^p dm.
\]
The case when $p = 1$ is similar. Since $\mu_u$ is the weak-star limit of a sequence $u_{r_n} dm$, it follows that $\|\mu_u\| \leq \|u\|_1$. On the other hand, by Fubini’s theorem
\[
\int_0^{2\pi} |u(re^{it})| \frac{dt}{2\pi} \leq \int_0^{2\pi} \int_{\mathbb{T}} P_z(\zeta) d|\mu(\zeta)| dm(\zeta) \frac{dt}{2\pi} = \|\mu_u\|.
\]
(ii) assume $(u_n)$ is a Cauchy sequence in $h^p$, and use (i) to conclude that the corresponding sequence of boundary functions, or measures is Cauchy in $L^p(m)$, respectively in the space of finite Borel measures on $\mathbb{T}$. Since these are complete spaces, we have that $(u_n)$ converges to $u$ in $L^p(m)$, or $(\mu_{u_n})$ converges in norm to $\mu$ and again by (i), $(u_n)$ converges in $h^p$ to the Poisson integral of $u$, or $\mu$. (iii) Almost everywhere on $\mathbb{T}$ we have
\[
|u_r(\zeta) - u(\zeta)| \leq u^*(\zeta), \quad \lim_{r \to 1^-} u_r(\zeta) = u(\zeta),
\]
and the result follows by Theorem 3.2 together with the dominated convergence theorem.

\[\square\]

**Exercise 1** Show that part (iii) of Theorem 5.1 fails for $p = \infty$ whenever $u$ has a discontinuous boundary function, or when $p = 1$, and $u$ is the Poisson integral of a measure with a nonzero singular part.

In the above proof we have used the important estimate
\[
|u(z)|^p \leq \int_{\mathbb{T}} P_z(\zeta) |u(\zeta)|^p dm(\zeta),
\]
6. Embeddings of $h^p$ into $L^p(\nu)$

This is an important problem with far-reaching consequences. We want to characterize the positive Borel measures $\nu$ on $D$ with the property that $h^p \subset L^p(\nu)$. An application of the closed graph theorem shows that if this inclusion holds, then the inclusion map must be bounded, that is

$$h^p \subset L^p(\nu) \iff \|u\|_p \leq C\|u\|_{L^p(\nu)},$$

for some $C > 0$ and all $u \in h^p$. Indeed, if $(u_n)$ converges to $u$ in $h^p$ and to $v$ in $L^p(\nu)$, then by Corollary 5.1, for each $z \in D$, $(u_n(z))$ converges to $u(z)$. Also, there exists a subsequence $(u_{n_k})$ which converges $\nu$-a.e. to $v$, hence $u = v \nu$-a.e.. For fixed $\lambda \in \mathbb{D}$, let

$$u_\lambda(z) = \frac{(1 - |\lambda|^2)^{1/p}}{(1 - \overline{\lambda}z)^{2/p}}, \quad z \in \mathbb{D}.$$ 

$u_\lambda$ is analytic, hence harmonic in $D$ and for $\zeta \in T$ we have $|u_\lambda(\zeta)| = P_\lambda(\zeta)$. Then the above inequality yields the following necessary condition for the inclusion $h^p \subset L^p(\nu)$

$$\sup_{\lambda \in \mathbb{D}} (1 - |\lambda|^2) \int |1 - \overline{\lambda}z|^{-2} d\nu(z) < \infty.$$
A similar but unfortunately stronger condition is sufficient and follows immediately from the estimates (5.8) and (5.7), namely

\[(6.10) \sup_{\zeta \in \mathcal{T}} \int \frac{1 - |z|^2}{|1 - \zeta z|^2} d\nu(z) < \infty.\]

For example, (5.7) and Fubini’s theorem give

\[
\int |u|^p d\nu \leq \int \int_{\mathcal{T}} P_z(\zeta)|u(\zeta)|^p dm(\zeta)d\nu(z) = \int_{\mathcal{T}} |u(\zeta)|^p \int \frac{1 - |z|^2}{|1 - \zeta z|^2} d\nu(z) \leq \|u\|_p^p \sup_{\zeta \in \mathcal{T}} \int \frac{1 - |z|^2}{|1 - \zeta z|^2} d\nu(z).
\]

**Exercise 1** Show that if \(\nu\) is the one-dimensional Lebesgue measure on the segment \((-1,1)\) then (6.9) holds, but (6.10) fails.

It turns out that the necessary condition above is also sufficient for our embedding. Let us rewrite this condition in a more transparent way.

**Proposition 6.1.** Given \(0 < h < 1\), and \(\theta \in [0, 2\pi]\), let

\[S_h(e^{i\theta}) = \{re^{it} : 1 - h < r < 1, |t - \theta| < h\}.\]

A positive Borel measure \(\nu\) on \(\mathbb{D}\) satisfies (6.9) if and only if

\[\sup_{0 < h < 1} \frac{\nu(S_h(e^{i\theta}))}{h} < \infty, \quad \text{or} \quad \sup_{\zeta \in \mathcal{T}, r > 0} \frac{\nu(\{|z - \zeta| < r\})}{r} < \infty.\]

**Proof.** We leave it as an exercise to show that if one supremum above is finite, then so is the other. Fix \(\lambda \in \mathbb{D} \setminus \{0\}\) and note that if \(|z - \frac{\lambda}{|\lambda|}| < (1 - |\lambda|)\), then

\[|1 - \overline{\lambda}z| \leq |\lambda||z - \frac{\lambda}{|\lambda|}| + 1 - |\lambda| < (|\lambda| + 1)(1 - |\lambda|),\]

hence

\[(1 - |\lambda|^2) \int |1 - \overline{\lambda}z|^{-2} d\nu(z) > \frac{(1 - |\lambda|^2)\nu(\{|z - \frac{\lambda}{|\lambda|}| < (1 - |\lambda|)\})}{4(1 - |\lambda|)^2} \geq \frac{\nu(\{|z - \frac{\lambda}{|\lambda|}| < (1 - |\lambda|)\})}{4(1 - |\lambda|)},\]
which shows that if (6.9) holds, then the second supremum above is finite. Conversely, fix \( \lambda \in \mathbb{D} \) with \(|\lambda| \geq 1/2\), and let

\[ E_n = \{ z \in \mathbb{D} : 2^n(1 - |\lambda|^2) \leq |1 - \overline{\lambda}z| < 2^{n+1}(1 - |\lambda|^2) \}. \]

Then

\[ (1 - |\lambda|^2) \int |1 - \overline{\lambda}z|^{-2}d\nu(z) \leq \sum_n \frac{\nu(E_n)}{2^{2n}(1 - |\lambda|^2)}. \]

Moreover, if \( z \in E_n \), then

\[ \frac{1}{2}|z - \frac{\lambda}{|\lambda|}| < |1 - \overline{\lambda}z| + 1 - |\lambda| < 2^{n+2}(1 - |\lambda|^2), \]

hence,

\[ \nu(E_n) \leq 2^{n+2}(1 - |\lambda|^2) \sup_{\zeta \in \mathbb{T}} \nu(\{|z - \zeta| < r\}) \frac{\nu(\{|z - \zeta| < r\})}{r}, \]

which gives

\[ (1 - |\lambda|^2) \int |1 - \overline{\lambda}z|^{-2}d\nu(z) \leq C \sup_{\zeta \in \mathbb{T}} \nu(\{|z - \zeta| < r\}) r, \]

when \( \lambda \in \mathbb{D} \) with \(|\lambda| \geq 1/2\). Since the integrals on the left are obviously bounded or \( |\lambda| < 1/2 \), the result follows.

The sets \( S_h \) have been used by Carleson who was the first to study embeddings of this type. They will be called Carleson boxes, and the measures satisfying one of the conditions in Proposition 6.1 are called Carleson measures. The next theorem is essentially due to Carleson.

**Theorem 6.1.** If \( \nu \) is a finite positive Borel measure on \( \mathbb{D} \), and \( 1 < p < \infty \) then \( h^p \subset L^p(\nu) \) if and only if \( \nu \) is a Carleson measure.

**Proof.** The necessity part has already been proved. Let \( u \in h^p \), \( \|u\|_p = 1 \). We claim that there exists \( k > 0 \) such that if \( t > 0 \) is sufficiently large then

\[ \nu(\{z \in \mathbb{D} : |u(z)| > t\}) \leq km(\{\zeta \in \mathbb{T} : u^*(\zeta) > t\}). \]

If this is achieved, it follows that

\[ \int |u|^p d\nu = p \int_0^\infty t^{p-1} \nu(\{z \in \mathbb{D} : |u(z)| > t\}) dt \]

\[ \leq kp \int_0^\infty t^{p-1} m(\{\zeta \in \mathbb{T} : u^*(\zeta) > t\}) dt \]

\[ = k \int_{\mathbb{T}} (u^*)^p dm, \]
hence, by Theorem 3.2, we have \( u \in L^p(\nu) \).

To see the claim, let \( \varepsilon > 0 \), and let \( G \) be an open subset of \( \mathbb{T} \) such that
\[
\{ \zeta \in \mathbb{T} : u^*(\zeta) > t \} \subset G, \quad m(G) < m(\{ \zeta \in \mathbb{T} : u^*(\zeta) > t \}) + \varepsilon.
\]

If \( z \in \mathbb{D} \setminus \{0\} \) is such that \( |u(z)| > t \) then \( z \) lies outside any Stolz angle \( \Gamma_\sigma(\zeta) \) (here \( \sigma \) is fixed), with \( u^*(\zeta) \leq t \). In particular, \( \frac{z}{|z|} \in G \). Recall that \( G \) is a countable union of disjoint open arcs \( I_n, n \geq 1 \) so that, if
\[
A_t = \{ z \in \mathbb{D} : |u(z)| > t \}, \quad A_{t,n} = \{ z \in \mathbb{D} : |u(z)| > t, \frac{z}{|z|} \in I_n \},
\]
we have
\[
A_t = \bigcup_{n \geq 1} A_{t,n}.
\]

Moreover, if \( z \in A_{t,n} \), and \( \zeta_n \) is the endpoint of \( I_n \) closest to \( z \) then
\[
|z - \zeta_n| < \frac{m(I_n)}{2} + 1 - |z|.
\]

Also, recall that \( z \notin \Gamma_\sigma(\zeta_n) \). By Corollary 5.1 it follows that if \( t \) is sufficiently large, then \( |z| > \sigma \), so there exists \( c > 1 \), depending only on \( \sigma \), with \( |z - \zeta| > c(1 - |z|) \). By replacing this in the last estimate we obtain
\[
1 - |z| < \frac{m(I_n)}{2} (c - 1), \quad z \in A_{t,n}.
\]

Then if \( m(G) \) is sufficiently small, which happens when \( t \) is large and \( \varepsilon \) is small, we can conclude that
\[
A_{t,n} \subset S_{h_n}(\zeta'_n),
\]
where \( \zeta'_n \) is the midpoint of \( I_n \), and \( h_n = c'm(I_n) \), for some fixed \( c' > 0 \). Then by the assumption on \( \nu \) we have
\[
\nu(A_{t,n}) \leq C m(I_n),
\]
and
\[
m(A_t) = \sum_{n \geq 1} m(A_{t,n}) \leq C (m(G)) < m(\{ \zeta \in \mathbb{T} : u^*(\zeta) > t \}) + \varepsilon,
\]
which implies the claim and completes the proof of the theorem.

**Example 6.1.** As pointed out in Exercise 1, the one-dimensional Lebesgue measure on the segment \((-1, 1)\) is a Carleson measure, that is, there exists \( C > 0 \) such that
\[
\int_{-1}^{1} |u(x)|^p dx \leq C\|u\|_p^p,
\]
for all $u \in h^p$, $1 < p \leq \infty$. If we consider only analytic functions the best constant is $C = \frac{1}{2}$ and the estimate is called the Fejér-Riesz inequality.

**Remark 6.1.** It is important to note that for a finite positive Borel measure $\nu$ the quantities considered in Proposition 6.1 are essentially related to the behavior of $\mu$ near the boundary. In fact, it is an immediate consequence of Corollary 5.1 that $\nu$ is a Carleson measure if and only if its restriction to any annulus $\{z : 0 < r < |z| < 1\}$ is a Carleson measure.

An important class of examples of Carleson measures are those of the form

$$d\nu = |\nabla u|^2(z) \log \frac{1}{|z|} \, dA,$$

where $u \in h^\infty$, $A$ is the normalized 2-dimensional Lebesgue measure restricted to $\mathbb{D}$, $dA = \frac{dx dy}{\pi}$, and $\nabla u$ stands for the gradient of $u$, so that

$$|\nabla u|^2 = \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial y} \right|^2.$$

This is a consequence of a more general result based on the following identity of Littlewood and Paley.

**Proposition 6.2.** Let $u \in h^2$ and $\lambda \in \mathbb{D}$. Then

$$(6.11) \int_T |u(\zeta) - u(\lambda)|^2 P_\lambda(\zeta) dm(\zeta) = \int_\mathbb{D} |\nabla u|^2(z) \log \frac{1 - \overline{\lambda} z}{|z - \lambda|} \, dA(z).$$

**Proof.** The proof is essentially based on the following direct application of Green’s formula: If $\varphi : \overline{\mathbb{D}} \to \mathbb{C}$ is twice continuously differentiable then

$$(6.12) \int_T \varphi dm = \varphi(0) + \int_\mathbb{D} \Delta \varphi(z) \log \frac{1}{|z|} \, dA(z).$$

Moreover, another direct computation shows that if $v$ is harmonic in $\mathbb{D}$, and $\varphi(z) = |v(z)|^2$, then

$$\Delta \varphi(z) = |\nabla v|^2(z).$$

Assume first that $u$ is harmonic in a larger disc, and set $v = u - u(0)$, $\varphi = |v|^2$. Then (6.12) immediately implies (6.11) when $\lambda = 0$. Thus, for any $u \in h^2$ and $0 < r < 1$, (6.11) holds for the dilation $u_r(z) = u(rz)$ and $\lambda = 0$. Also, if $0 < r, \rho < 1$ we can apply (6.11) to obtain

$$\|u_r - u_\rho\|^2_2 = \int_\mathbb{D} |\nabla u_r(z) - \nabla u_\rho(z)|^2 \log \frac{1}{|z|} \, dA(z),$$
and by Theorem 5.1 (iii) it follows that $\nabla u_r - \nabla u_\rho$ converges to zero in $L^2(\mathbb{D}, \log \frac{1}{|z|} dA)$, when $r, \rho \to 1^-$. This easily shows that when $\lambda = 0$, (6.11) holds for every $u \in h^2$.

If $\lambda \in \mathbb{D}$ is arbitrary, let $\phi_\lambda(z) = \frac{z-\lambda}{1-\lambda z}$. Clearly, $\phi_\lambda$ is an analytic bijection from $\mathbb{D}$ onto itself. By what we have proved above, we have for every $u \in h^2$,

$$\int_T |u \circ \phi_\lambda - u(\lambda)|^2 dm = \int_\mathbb{D} |\nabla u \circ \phi_\lambda|^2(z) \log \frac{1}{|z|} dA(z).$$

A direct computation similar to the above yields

$$|\nabla u \circ \phi_\lambda|^2(z) = |\nabla u|^2(\phi_\lambda(z)) |\phi'_\lambda(z)|^2,$$

i.e.

$$\int_T |u \circ \phi_\lambda - u(\lambda)|^2 dm = \int_\mathbb{D} |\nabla u|^2(\phi_\lambda(z)) |\phi'_\lambda(z)|^2 \log \frac{1}{|z|} dA(z).$$

Then (6.11) follows directly from the change of variables $\xi = \phi_\lambda(\zeta)$, $w = \phi_\lambda(z)$. $\square$

**Exercise 1** Prove (6.12).

**Corollary 6.1.** Let $u \in h^2$. Then the measure $|\nabla u|^2(z) \log \frac{1}{|z|} dA$, is a Carleson measure if and only if

$$\sup_{\lambda \in \mathbb{D}} \int_T |u(\zeta) - u(\lambda)|^2 P_\lambda(\zeta) dm(\zeta) < \infty.$$  

**Proof.** If the condition in the statement holds, we apply Proposition 6.2 together with the inequality $\log \frac{1}{x} \geq 1 - x$, $0 < x \leq 1$, to obtain

$$(6.13) \sup_{\lambda \in \mathbb{D}} \int_\mathbb{D} |\nabla u|^2(z) \frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\overline{\lambda}z|^2} dA(z) < \infty,$$

that is, (6.9) holds for the measure $|\nabla u|^2(z)(1 - |z|^2)dA$. Obviously, this measure is comparable to the measure $|\nabla u|^2(z) \log \frac{1}{|z|} dA$ outside the disc centered at the origin and of radius $\frac{1}{2}$, hence by Remark 6.1, $|\nabla u|^2(z) \log \frac{1}{|z|} dA$, is a Carleson measure. To see the converse, use again the inequality $\log \frac{1}{x} \geq 1 - x$, $0 < x \leq 1$, to conclude that $|\nabla u|^2(z)(1 - |z|^2)dA$ is a Carleson measure, that is, (6.13) holds. We want to prove that

$$\sup_{\lambda \in \mathbb{D}} \int_\mathbb{D} |\nabla u|^2(z) \log \frac{1 - \overline{\lambda}z}{|z - \lambda|} dA(z) < \infty.$$
After the change of variables \( w = \phi_\lambda(z) \) in both integrals above, we see that it suffices to verify the estimate

\[
\int_D |\nabla u|^2(z) \log \frac{1}{|z|} dA(z) \leq C \int_D |\nabla u|^2(z)(1 - |z|^2)dA(z),
\]

for some \( C > 0 \) and all \( u \in h^2 \). This is a simple exercise based on Parseval’s formula and is left to the reader. □

Obviously, bounded harmonic functions have the property stated in Corollary 6.1, but there are also unbounded functions with this property. An easy example whose verification is left to the reader is the analytic function

\[
f(z) = \log(1 - z), \quad z \in \mathbb{D}.
\]

There is an important intrinsic characterization of these functions in terms of the boundary values alone. The Poisson integral \( u \) of a function \( h \in L^1(m) \) satisfies the conditions in Corollary ?? if and only if

\[
\sup_I \frac{1}{m(I)} \int_I \left| h - \frac{1}{m(I)} \int_I h dm \right| dm < \infty,
\]

where the supremum is taken over all arcs \( I \subset \mathbb{T} \). We say that such a function has bounded mean oscillation and denote the space of all such functions by \( BMO \).

The proof of the above assertions relies on a real-variable argument which will not be covered here.
CHAPTER 3

Hardy spaces

The Hardy space $H^p$, $0 < p \leq \infty$, is the subspace of $h^p$ consisting of analytic functions in $\mathbb{D}$. A direct application of Corollary 5.1 shows that when $1 \leq p \leq \infty$, $H^p$ is a closed subspace of $h^p$, in particular, it is a Banach space. These spaces are the object of our study, and some of the tools are provided by the material presented in the previous chapter.

1. Outer functions.

In this section we shall apply the results on Poisson integrals in order to construct analytic functions with prescribed boundary values of their modulus.

**Proposition 1.1.** Let $h$ be a nonnegative function on the unit circle such that

$$\int_{\mathbb{T}} |\log h| dm < \infty.$$  

Then function $F$ defined in $\mathbb{D}$ by

$$F(z) = \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta) dm(\zeta) \right)$$

has nontangential limits almost everywhere on the unit circle. Moreover, if we denote by $F(\zeta)$ the nontangential limit of $F$ at $\zeta \in \mathbb{T}$ (whenever it exists) then $|F(\zeta)| = h(\zeta)$ m−a.e. on $\mathbb{T}$.

**Proof.** We leave it as an exercise to prove that $F$ is analytic. Also note that this function satisfies

$$|F(z)| = \exp \text{Re} \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta) dm(\zeta) \right) = \exp \left( \int_{\mathbb{T}} P_z(\zeta) \log h(\zeta) dm(\zeta) \right).$$

Consequently, by the results about Poisson integrals, $|F|$ has the nontangential limit $\exp \log h(\zeta) = h(\zeta)$, m−a.e.. It remains to show that $F$ has nontangential limits almost everywhere as well. This follows from
the simple observation that if $h$ is bounded above then the function $F$ is bounded in $\mathbb{D}$. Indeed, if $h \leq M$ then

$$|F(z)| = \exp \left( \int_{\mathbb{T}} P_z(\zeta) \log h(\zeta) d\mu(\zeta) \right) \leq \exp \left( \int_{\mathbb{T}} P_z(\zeta) \log M d\mu(\zeta) \right) = M.$$  

This means that the functions $F_1, F_2$ defined by

$$F_1(z) = \exp \left( \int_{\mathbb{T}} \frac{z + \zeta}{\zeta - z} \log \min \{h(\zeta), 1\} d\mu(\zeta) \right),$$

and

$$F_1(z) = \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \max \{h(\zeta), 1\} d\mu(\zeta) \right),$$

are analytic and bounded in $\mathbb{D}$, hence both functions have nontangential limits almost everywhere on the unit circle. By the above computation we also know that these limits are nonzero $m$-a.e.. Thus $F = F_1 / F_2$ $F$ has nontangential limits almost everywhere on the unit circle as well. \hfill \square

Of course, the function $F$ constructed above, is not the unique analytic functions such that the modulus of its nontangential limits take the prescribed value $h$ a.e. If $B$ is any finite Blaschke product then $|B|$ extends continuously to the boundary and takes the value $1$ there. Thus, $BF$ shares the same property as $F$. It is also possible to construct another analytic function $G$ in $\mathbb{D}$ such that $|G|$ has the same nontangential limits as $|F|$ a.e. and in addition, $G$ has no zeros in $\mathbb{D}$. Indeed, if $g(z) = \exp(-\frac{1}{1+z})$ then $G = gF$ has the required property because $g$ extends continuously to $\partial \mathbb{D} \setminus \{1\}$ with $|g(\zeta)| = 1$ for all $\zeta \in \partial \mathbb{D} \setminus \{1\}$. Nevertheless, the function $F$ constructed in the proof of the above theorem plays a central role in what follows.

**Definition 1.1.** An analytic function $F$ in $\mathbb{D}$ is called an outer function if there exists a nonnegative function $h$ on the unit circle such that

$$\int_{\mathbb{T}} |\log h| d\mu < \infty$$

and

$$F(z) = \exp \left( \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta) d\mu(\zeta) \right).$$

In this case, $F$ is the outer function whose modulus equals $h$ a.e. on the boundary.
Exercise 1. Show that the product and quotient of two outer functions is outer. Moreover, prove that if \( F \) is outer with \( |F(e^{it})| \geq a > 0 \) a.e. then \( |F(z)| \geq a \) for all \( z \in \mathbb{D} \). Finally, show that any function \( F \) that is analytic in a larger disc (than \( \mathbb{D} \)) and has no zeros in \( \mathbb{D} \) is an outer function, but the statement is no longer true if we replace the assumption "\( F \) analytic in a larger disc" by "\( F \) continuous in \( \mathbb{D} \)."

Proposition 1.2. Let \( h \) be a nonnegative function on the unit circle such that \( \int_T |\log h| dm < \infty \) and let \( F_h \) be the outer function whose modulus equals \( h \) a.e. on the boundary. Then
\[
|F_h(z)| \leq \int_T P_z(\zeta)h(\zeta)dm(\zeta).
\]

The proof is essentially based on the following simple lemma.

Lemma 1.1. If \( u, v : [a, b] \to \mathbb{R} \) are integrable functions, \( v \) is nonnegative with \( \int_a^b v(x)dx = 1 \) then
\[
e^{\int_a^b u(t)v(t)dt} \leq \int_a^b e^{u(t)}v(t)dt.
\]

Proof. The inequality is equivalent to
\[
1 \leq \int_a^b v(t) \left( e^{u(t)} - \int_a^b u(x)v(x)dx \right) dt.
\]
Since for all \( T \in \mathbb{R} \) we have \( e^T \geq T + 1 \), we deduce that
\[
e^{u(t)} - \int_a^b u(x)v(x)dx \geq u(t) - \int_a^b u(x)v(x)dx + 1 , \quad t \in [a, b].
\]
Integration on \([a, b]\) (w.r.t \( t \)) now yields
\[
\int_a^b v(t) \left( e^{u(t)} - \int_a^b u(x)v(x)dx \right) dt \geq \int_a^b v(t) \left( u(t) - \int_a^b u(x)v(x)dx + 1 \right) dt = 1
\]
and the result follows. \( \square \)

Exercise 2. Prove the following generalization of Lemma 1.1 which is called Jensen’s inequality.

Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a differentiable convex function, that is, \( \phi \) satisfies
\[
\phi(x) - \phi(y) \geq \phi'(y)(x - y), \quad x, y \in \mathbb{R}.
\]
If \( u, v : [a, b] \to \mathbb{R} \) are integrable functions, with \( v \geq 0 \) and \( \int_a^b v(x)dx = 1 \) then
\[
\phi \left( \int_a^b u(t)v(t)dt \right) \leq \int_a^b \phi(u(t))v(t)dt.
\]

**Proof of Proposition 1.2.** Recall first that \( \int_T P_z dm = 1 \) for all \( z \in \mathbb{D} \) and set \( u(t) = \log h(e^{it}), \ v(t) = \frac{1}{2\pi} P_z(e^{it}). \) Then by Lemma 1.1 we obtain
\[
|F_h(z)| = \exp \left( \int_T P_z(\zeta) \log h(\zeta)dm(\zeta) \right) \leq \int_T P_z(\zeta)h(\zeta)dm(\zeta).
\]

**Corollary 1.1.** Let \( h \) be a nonnegative function on the unit circle such that
\[
\int_T |\log h|dm < \infty
\]
and let \( F_h \) be the outer function whose modulus equals \( h \) a.e. on the boundary. Then for \( 0 < p \leq \infty \), \( F_h \in H^p \) if and only if \( h \in L^p(m) \), and
\[
\|F_h\|_p = \|h\|_{L^p(m)}.
\]

**Proof.** When \( 1 < p \leq \infty \), the result follows from the properties of Poisson integrals. However, the following argument works for all \( p > 0 \). The inequality
\[
\|F_h\|_p \geq \|h\|_{L^p(m)},
\]
follows by an application of Fatou’s lemma and Proposition 1.1. The reverse inequality follows by Proposition 1.2 applied to the outer function \( F_h^p \), which is obviously well defined. \( \square \)

### 2. Zeros of \( H^p \)-functions

It turns out that in terms of zeros, the condition defining membership in \( H^p \) is quite restrictive, and in fact, it is equivalent to the Blaschke condition. This has a number of important consequences for the development of the theory.

We begin with a lemma on the integrals involved in the definition of \( H^p \) norms.

**Lemma 2.1.** If \( f \) is analytic in \( \mathbb{D} \) and \( 0 < p < \infty \), then
\[
M_p(r, f) = \int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi},
\]
is an increasing function of \( r \in [0, 1) \).
2. ZEROS OF $H^p$-FUNCTIONS

Proof. If $p > 1$, the result can be obtained using properties of Poisson integrals. The argument below is valid for all values $p > 0$. It is easy to verify that for $0 < r < 1$,

$$\int_T |\log |f(r\zeta)|| dm(\zeta) < \infty,$$

so that we can consider the outer function $F_r$ with $|F_r(\zeta)| = |f(r\zeta)|$, $m$-a.e. on $T$. Moreover, if $f$ has no zeros on $rT$, then $|F_r|$ is bounded below, hence $f_r/F_r$ is bounded in $\mathbb{D}$, where, as usual $f_r(z) = f(rz)$, $z \in \mathbb{D}$. Since the nontangential limits of $f_r/F_r$ have modulus one a.e. on the circle, it follows by the Poisson representation that $|f_r/F_r| \leq 1$, that is,

$$|F_r(z)| \geq |f(rz)|, \quad z \in \mathbb{D}.$$  

Then for $0 < \rho < 1$ we have by Corollary 1.1

$$M_p(r\rho, f) \leq \|f_r\|_p^p \leq \|F_r\|_p^p = M_p(r, f).$$

Thus we have shown the inequality $M_p(r\rho, f) \leq M_p(r, f)$ for all $\rho \in (0, 1)$ and all $r$ in a dense subset of $[0, 1]$). The result follows. \qed

Theorem 2.1. Let $0 < p \leq \infty$, and assume that $f \in H^p$ is not identically zero. Let $a_1, a_2, \ldots$ be the zeros of $f$ in $\mathbb{D}$, repeated according to their multiplicity. Then

$$\sum_{n \geq 1} (1 - |a_n|) < \infty.$$  

Moreover, if $B$ is the Blaschke product with zeros $a_1, a_2, \ldots$ then $f/B \in H^p$, and

$$\|f/B\|_p = \|f\|_p.$$

Proof. Denote by $B_N$ the partial products

$$B_N(z) = z^n \prod_{n=m+1}^N \frac{-\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z}.$$  

Then $f/B_N$ is analytic in $\mathbb{D}$. We claim that $f/B_N \in H^p$ with

$$\|f/B_N\|_p = \|f\|_p.$$  

If $p = \infty$ this follows immediately from the maximum principle together with the fact that $B_N$ is continuous in the closed unit disc and $|B_N| = 1$ on $T$. Indeed, if we pick a a sequence $(z_n)$, $|z_n| \to 1$ such that

$$\|f/B_N\|_\infty = \lim_{n \to \infty} \frac{|f(z_n)|}{|B_N(z_n)|}.$$
from $|B_N(z_n)| \to 1$, $n \to \infty$, we obtain

$$||f/B_N||_\infty = \lim_{n \to \infty} |f(z_n)| \leq ||f||_\infty.$$  

The reverse inequality is obvious, since $|B_N| \leq 1$ in $\mathbb{D}$. The case when $0 < p < \infty$ is similar, but the maximum principle is replaced by Lemma 2.1. By this result we have

$$\|f/B_N\|^p_p = \lim_{r \to 1^-} M_p(r, f/B_N) \leq \limsup_{r \to 1^-} M_p(r, f) \max_{|z|=r} (|B_N(z)|^{-p} = \|f\|^p_p,$$

and, again the reverse inequality is immediate from $|B_N| \leq 1$ in $\mathbb{D}$. With the claim in hand, we see that $(B_N)$ cannot converge to zero uniformly on compact subsets of $\mathbb{D}$, since $f$ does not vanish identically, hence this would contradict the estimates

$$M_p(r, f/B_N) \leq \|f\|^p_p, \quad \max_{|z|=r} |f/B_N(z)| \leq \|f\|_\infty,$$

valid for all $r \in (0, 1)$. Thus the Blaschke condition in the statement holds true. Letting $N \to \infty$, we obtain from the last estimates

$$M_p(r, f/B) \leq \|f\|^p_p, \quad \max_{|z|=r} |f/B(z)| \leq \|f\|_\infty,$$

e. $f/B \in H^p$ and $\|f/B\|^p_p \leq \|f\|^p_p$. The reverse inequality follows again from $|b(z)| < 1$, $z \in \mathbb{D}$, and the proof is complete. $\Box$

**Exercise 1.** In the case when $p = \infty$ Theorem 2.1 belongs to a very classical circle of ideas which includes the famous Schwarz Lemma. Prove the following assertions:

(i) *(Schwarz Lemma)* Let $f$ be analytic in $\mathbb{D}$ and satisfy $|f(z)| \leq 1$ there. If $f(0) = 0$ then

$$|f(z)| \leq |z|, \quad z \in \mathbb{D},$$

and

$$|f'(0)| \leq 1.$$  

Moreover, if equality occurs in one of the above inequalities then $f$ is a rotation of the unit disc, i.e. $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

(ii) *(Schwarz-Pick Lemma)* Let $f$ be analytic in $\mathbb{D}$ and satisfy $|f(z)| \leq 1$ there. Then if $z, w \in \mathbb{D}$, $z \neq w$ we have

$$\frac{|f(z) - f(w)|}{|1 - f(w)f(z)|} \leq \frac{|z - w|}{|1 - wz|}.$$
with equality if and only if $f$ has the form $f(z) = \alpha \frac{z-a}{1-\overline{a}z}$ for some fixed $a \in \mathbb{D}$ and $\alpha \in \partial \mathbb{D}$. In particular, for all $z \in \mathbb{D}$ we have also

$$|f'(z)| \leq \frac{1}{1-|z|^2}.$$ 

**Corollary 2.1.** Let $(a_n)$ be a sequence in $\mathbb{D}$ such that

$$\sum_{n \geq 1} (1 - |a_n|) < \infty$$

and let $B$ be the corresponding Blaschke product. Then $B$ has nontangential limits of modulus one almost everywhere on the unit circle.

**Proof.** Clearly $B$ is in $H^\infty$ with $\|B\|_\infty \leq 1$. Let $f$ be any bounded analytic function in $\mathbb{D}$ and apply Theorem 2.1 to the function $Bf$. Then the conclusion reads

$$\|fB/B\|_\infty = \|f\|_\infty = \|Bf\|_\infty.$$ 

Now suppose there exist $\varepsilon > 0$ and a measurable set $E \subset \mathbb{T}$, $m(E) > 0$ such that the nontangential limits $B(\zeta)$ satisfy $|B(\zeta)| < 1 - \varepsilon$ for $\zeta \in E$. Let $f$ be the outer function whose modulus equals 1 a.e. on $E$ and $\frac{1}{2}$ a.e. on its complement. Then by Corollary 5.3, Chapter I we have that $\|f\|_\infty = 1$, but

$$\|Bf\|_\infty = \text{essup}\{|Bf| < \max\{\frac{1}{2}, 1 - \varepsilon\} < 1$$

which contradicts the previous equality. Thus, for all $\varepsilon > 0$ the set of points $\zeta$ where $|B(\zeta)| < 1 - \varepsilon$ has measure zero and hence, the set

$$\{\zeta, \ |B(\zeta)| < 1\} = \bigcup_{n=1}^{\infty} \{\zeta, \ |B(\zeta)| < 1 - 1/n\}$$

has measure zero as well. \qed

3. Applications

Theorem 2.1 yields a very useful factorization of $H^p$-functions, which has a number of important consequences. According to this result, every nonzero $f \in H^p, \ 0 < p \leq \infty$ can be written in the form

$$f =Bg,$$

where $B$ is a Blaschke product and $g \in H^p$ is zero-free in $\mathbb{D}$ with $\|g\|_p = \|f\|_p$. 

3.1. Boundary behavior. With the above factorization in hand, we can easily apply the results obtained for Poisson integrals in Section 5 of the previous chapter. Indeed, if \( f = Bg \in H^p \), the zero-free factor \( g \) belongs to \( H^p \) so that, \( g^{p/2} \) belongs to \( H^2 \subset h^2 \). This simple observation leads to a quick proof of the following theorem.

**Theorem 3.1.** (i) If \( f \in H^p \), \( 0 < p \leq \infty \), then \( f \) has nontangential limits \( f(\zeta) \) for \( m \)-a.e. \( \zeta \in \mathbb{T} \) and
\[
\|f\|_p = \|f\|_{L^p(m)}.
\]
(ii) If \( f \in H^p \), \( 0 < p \leq \infty \), then its maximal nontangential function \( f^* \) belongs to \( L^p(m) \), and there exists \( c_p > 0 \) such that
\[
\|f^*\|_{L^p(m)} \leq C_p \|f\|_p.
\]

**Proof.** (i) As pointed out above, if \( f = Bg \) with \( B \) a Blachke product and \( g \in H^p \) zero-free in \( \mathbb{D} \), then \( g^{p/2} \in H^2 \), which implies that \( g^{p/2} \) has nontangential limits \( m \)-a.e.. Since \( B \in H^\infty \), \( f = BG \) has nontangential limits \( m \)-a.e.. By Corollary 2.1 we have \( |f| = |g|, m \)-a.e. on \( \mathbb{T} \), and using also Theorem 5.1 we obtain
\[
\|f\|_p = \|g\|_p = \|g^{p/2}\|^{2/p}_2 = \|g^{p/2}\|_{L^2(m)}^{2/p} = \|f\|_{L^p(m)}.
\]

(ii) If \( f = Bg \) as above \( f^* \leq g^* = (g^{p/2})^* \), and the result follows again from the fact that \( g^{p/2} \in H^2 \). □

The norm convergence of dilations also extends for all values of \( 0 < p < \infty \) as the following result shows.

**Theorem 3.2.** If \( f \in H^p \), \( 0 < p < \infty \), and for \( 0 \leq r < 1 \), \( f_r(z) = f(rz) \), \( z \in \mathbb{D} \), then
\[
\lim_{r \to 1^-} \|f - f_r\|_p = 0.
\]

**Proof.** By Theorem 3.1 (i) we have
\[
\|f - f_r\|_p = \int_{\mathbb{T}} |f - f_r|^p dm,
\]
and since \( |f(\zeta) - f(r\zeta)| \leq 2f^*(\zeta), \zeta \in \mathbb{T} \), the result follows by an application of the dominated convergence theorem □

As pointed out in the previous chapter, the result fails when \( p = \infty \).

**Corollary 3.1.** (i) For \( 1 \leq p < \infty \), \( H^p \) is a separable Banach space.
(ii) For \( 0 < p < 1 \), \( H^p \) is a complete separable space w.r.t. the metric
\[
d_p(f, g) = \|f - g\|_p^p.
\]
Proof. If \( f \in H^p \), and \( 0 < r < 1 \), then \( f_r \) can be approximated uniformly by polynomials, so that polynomials with rational coefficients form a countable dense subset of \( H^p \). The verification of the remaining assertions in (ii) is straightforward.

3.2. Poisson representation of \( H^1 \) and the F. & M. Riesz theorem. Since \( H^p \subset h^p \), it follows that for \( 1 < p \leq \infty \), every function in \( H^p \) is the Poisson integral of its boundary function. Interesting enough, this result continues to hold for \( H^1 \) as well, even if it fails for the larger space \( h^1 \).

**Theorem 3.3.** If \( f \in H^1 \) then

\[
 f(z) = \int_T P_z(\zeta) f(\zeta) \, dm(\zeta), \quad \zeta \in \mathbb{D}.
\]

**Proof.** This is a direct application of Theorem 3.2. Since for \( 0 < r < 1 \) \( f_r \in H^\infty \), we have

\[
 f(rz) = \int_T P_z(\zeta) f(r\zeta) \, dm(\zeta), \quad \zeta \in \mathbb{D}.
\]

Let \( r \to 1^- \), and apply Theorem 3.2 together with the fact that for fixed \( z \in \mathbb{D} \), the Poisson kernel \( P_z(\cdot) \) belongs to \( L^\infty(m) \).

The next theorem is a famous result due to the brothers Riesz with wide applications in function theory. The original proof is quite different from the one below and is of interest in its own right.

**Theorem 3.4.** Let \( \mu \) be a finite Borel measure on \( \mathbb{T} \) with the property that

\[
 \int_T \zeta^n d\mu(\zeta) = 0,
\]

for all nonnegative integers \( n \). Then \( \mu \) is absolutely continuous w.r.t. \( m \) and there exists \( f \in H^1 \) with \( f(0) = 0 \) such that \( \frac{d\mu}{dm} = f \).

**Proof.** Let \( u \) be the Poisson integral of \( \mu \), and recall that \( u \) determines \( \mu \) uniquely. Clearly, \( u \in h^1 \), and \( u(0) = 0 \). We show that \( u \) is analytic in \( \mathbb{D} \), hence \( u \in H^1 \). For \( z \in \mathbb{D} \),

\[
 2u(z) = 2 \int_T P_z(\zeta) d\mu(\zeta) = \int_T \frac{\bar{\zeta} + \bar{z}}{\zeta - z} d\mu(\zeta) + \int_T \frac{\zeta + z}{\zeta - z} d\mu(\zeta).
\]

If \( \zeta \in \mathbb{T} \)

\[
 \frac{\bar{\zeta} + \bar{z}}{\zeta - z} = \frac{1 + \bar{z}\zeta}{1 - \bar{z}\zeta} = 1 + 2 \sum_{n=1}^{\infty} \bar{z}^n \zeta^n,
\]

\[
 2u(z) = \int_T \left( \frac{1 + \bar{z}\zeta}{1 - \bar{z}\zeta} + \frac{\zeta + z}{\zeta - z} \right) d\mu(\zeta).
\]
and the series converges uniformly on $\mathbb{T}$, for fixed $z \in \mathbb{D}$. Thus by assumption,
\[
\int_{\mathbb{T}} \frac{\zeta + \bar{z}}{\zeta - z} d\mu(\zeta) = 0,
\]
and
\[
2u(z) = \text{int}_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta),
\]
so that $u \in H^1$. Then the result follows by Theorem 3.3. \hfill \Box

### 3.3. Inner-outer factorization.

The factorization used in the previous section can be further refined. The crucial step is to construct for nonzero $f \in H^p$ the outer function $F|f|$. To this end we need to show that $\log |f| \in L^1(m)$, a quite remarkable property of $H^p$-functions.

The classical approach to this fact is via the so-called Jensen's formula, but we shall follow a different path here and begin with the following inequality, which is an analogue of (5.7), Chapter 2.

**Proposition 3.1.** If $f \in H^p$, $0 < p \leq \infty$, then for all $z \in \mathbb{D},$
\[
|f(z)|^p \leq \int_{\mathbb{T}} P_z(\zeta)|f(\zeta)|^p dm(\zeta).
\]

In particular,
\[
|f(z)| \leq \left( \frac{1 + |z|}{1 - |z|} \right)^{1/p} \|f\|_p, \quad z \in \mathbb{D}.
\]

**Proof.** Write again $f = Bg$, as in the previous proofs, with $B$ a Blachke product and $g \in H^p$ zero-free in $\mathbb{D}$, and recall that $|g| = |f|, \text{ m.a.e. on } \mathbb{T}$. Since $g^{p/2} \in H^2 \subset h^2$ we can apply the estimate (5.7) to obtain
\[
|f(z)|^p \leq |g^{p/2}(z)|^2 \leq \int_{\mathbb{T}} P_z(\zeta)|g(\zeta)|^p dm(\zeta) = \int_{\mathbb{T}} P_z(\zeta)|f(\zeta)|^p dm(\zeta).
\]

The second part follows by the standard estimate of the Poisson kernel. \hfill \Box

This inequality exhibits a harmonic majorant for $H^p$-functions, and it turns out that such harmonic majorants can be used to give an alternative definition of Hardy spaces which extends to more general plane domains as well. We shall not pursue this matter here.

**Exercise 1.** Show that if $f \in H^p$, $0 < p \leq \infty$, the harmonic function
\[
u_f(z) = \int_{\mathbb{T}} P_z(\zeta)|f(\zeta)|^p dm(\zeta)
\]
is the least harmonic majorant of \(|f|^p\) in \(\mathbb{D}\), that is, if \(v\) is harmonic in \(\mathbb{D}\) with \(v \geq |f|^p\), then \(v \geq u_f\).

With Proposition 3.1 in hand we can turn to our original goal.

**Proposition 3.2.** If \(f \in H^p\), \(0 < p \leq \infty\) is not identically zero then \(\log |f| \in L^1(m)\). Moreover, if \(F\) is the outer function with \(F = \log |f|\), \(m\)-a.e. on \(\mathbb{T}\) then

\[ |f(z)| \leq |F(z)|, \quad z \in \mathbb{D}. \]

**Proof.** We start with the case when \(p = \infty\) and assume without loss of generality that \(\|f\|_\infty \leq 1\). Write \(f = Bg\) with \(B\) a Blachke product and \(g \in H^p\) zero-free in \(\mathbb{D}\), and note that

\[ \log |f(\zeta)| = -\log |f(\zeta)| = -\log |g(\zeta)|, \quad m\text{-a.e. on } \mathbb{T}. \]

By Fatou’s lemma

\[ -\int_\mathbb{T} \log |g(\zeta)| dm(\zeta) \leq \liminf_{r \to 1^-} \int_\mathbb{T} \log |g(r\zeta)| dm(\zeta) = -\log |g(0)|, \]

because \(\log |g|\) is harmonic in \(\mathbb{D}\). If \(p \in (0, \infty)\), let \(G\) be the outer function with \(|G(\zeta)| = \exp(|f(\zeta)|^p)\), \(m\text{-a.e. on } \mathbb{T}\) and note that:

1) \(|G(z)| \geq 1\), \(z \in \mathbb{D}\),
2) \(|f/G|\) is essentially bounded on \(\mathbb{T}\) \((e^{r^p} > c_x)\).

From the first inequality we see that \(f/G \in H^p\) and if we apply Proposition 3.1 to \(f/G\) it follows easily that \(f/G \in H^\infty\). Then by the first part of the proof we have that

\[ \log |f| - \log |G| = \log |f| - |f|^p \in L^1(m), \]

i.e. \(\log |f| \in L^1(m)\). To see the second part of the statement, we consider the sequence of outer functions \((F_n)\) with \(|F_n(\zeta)| = \log(|f(\zeta)| + \frac{1}{n})\), \(m\text{-a.e. on } \mathbb{T}\). Exactly the same reasoning as before shows that \(f/F_n \in H^\infty\) with \(\|f/F_n\|_\infty \leq 1\). Using the first part of the proof we can easily show that \(F_n(z) \to F(z)\) for all \(z \in \mathbb{D}\), which gives the desired result. \(\square\)

The outer function \(F\) in Proposition 3.2 will be called the outer factor of \(f\).

**Definition 3.1.** A bounded analytic function \(I\) in \(\mathbb{D}\) is called \textit{inner} if its nontangential limits satisfy \(|I(\zeta)| = 1\), \(m\text{-a.e. on } \mathbb{T}\).

Clearly, inner functions are bounded by 1 in \(\mathbb{D}\). Note that unimodular constant are inner functions. Less trivial examples are provided by Blaschke products. According to Proposition 3.2 \(I = f/F\) is inner,
whenever $f \in H^p$. This function will be called the inner factor of $f \in H^p$. and the factorization

$$f = IF$$

will be referred to as the inner-outer factorization of $f$.

**Proposition 3.3.** The inner outer factorization (3.14) of $f \in H^p$, $0 < p \leq \infty$ is unique.

**Proof.** If $f = IF = JG$, with $I, J$ inner and $F, G$ outer then

$$\frac{I}{J} = \frac{G}{F},$$

which implies that the nontangential limits of $G/F$ are unimodular a.e. on $\mathbb{T}$. Since $G/F$ is outer it follows by definition that $G/F = 1$. 

At its turn, by Theorem 2.1 the inner factor can be further decomposed as $I = BS$, where $B$ is a Blaschke product and a function $S$ which is zero-free in $\mathbb{D}$. Here we consider $B = 1$ if $I$ has no zeros to begin with.

**Definition 3.2.** An inner function without zeros in $\mathbb{D}$ is called singular inner.

Singular inner functions have a very special form. If $S$ is such a function then $\log |S|$ is harmonic and negative in $\mathbb{D}$, in particular it belongs to $h^1$. Thus, by Corollary 2.2 in Chapter 2, there is a nonnegative finite Borel measure $\mu$ on $\mathbb{T}$ such that

$$\log |S(z)| = -\int_{\mathbb{T}} P_z(\zeta)d\mu(\zeta).$$

Since $\log |S|$ has nontangential limits zero a.e. on $\mathbb{T}$, it follows by Corollary 4.1 in Chapter 2, that the measure $\mu$ is singular w.r.t. $m$. This argument together with analytic completion and exponentiation yields the following representation formula for singular inner functions.

**Corollary 3.2.** If $S$ is a singular inner function then there exists $\alpha \in \mathbb{R}$ and a nonnegative finite Borel measure $\mu$ on $\mathbb{T}$ which is singular w.r.t. $m$, such that

$$S(z) = \exp \left(i\alpha - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}d\mu(\zeta) \right).$$

The above argument refines the inner outer factorization $f = IF$, of $f \in H^p$. The final result is stated below, but already proved above.
Theorem 3.5. Let $f \in H^p$, $0 < p \leq \infty$. If $f$ is not identically zero, it can be written uniquely in the form

$$f = BSF,$$

where $B$ is a Blaschke product (or $B = 1$), $S$ is a singular inner function, and $F$ is the outer factor of $f$.

Exercise 2. Show that if $f, 1/f \in H^p$ for some $p > 0$, then $f$ is outer.

Exercise 3. Show that a singular inner function cannot extend continuously to the closed unit disc unless it is constant. Construct a function that extends continuously to $D$ and yet has a nontrivial singular inner factor. Similarly one can construct such a function that has infinitely many zeros. Hint: Take the Blaschke product with zeros $1 - \frac{1}{n^2}$, and multiply by $(1 - z)$. Then show that the product is continuous.

Exercise 4. What is the canonical factorization given by Theorem 3.5 of the following functions?

a) $f(z) = (2z - 1)e^z$, $z \in \mathbb{D}$.

b) $g(z) = \exp(\frac{1}{z^2 - 1})$, $z \in \mathbb{D}$.

c) $h(z) = \sin z$, $z \in \mathbb{D}$.

3.4. Littlewood subordination. Given $f, g$ analytic in $\mathbb{D}$, we say that $f$ is subordinate to $g$, and write $f \prec g$ if there exists $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ analytic with $\varphi(0) = 0$ such that $f = g \circ \varphi$. Note that in this case $f(\mathbb{D}) \subseteq g(\mathbb{D})$, and by the Schwarz lemma we have $f(r\mathbb{D}) \subseteq g(r\mathbb{D})$, for all $r \in [0,1]$. J. E. Littlewood proved long ago that subordination decreases the $H^p$-norm, and his theorem is part of the following result.

Theorem 3.6. Let $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ be analytic. Then for every $f \in H^p$, $0 < p < \infty$, we have $f \circ \varphi \in H^p$ and

$$\|f \circ \varphi\|_p \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/p} \|f\|_p.$$

Equality holds for some $0 < p < \infty$ and all $f \in H^p$, if and only if $\varphi$ is inner with $\varphi(0) = 0$.

Proof. The norm inequality in the statement is a direct consequence of Proposition 3.1, since by that inequality we have for every $f \in H^p$,

$$|f(\varphi(z))|^p \leq \int_\pi P_{\varphi(z)}(\zeta)|f(\zeta)|^pdm(\zeta).$$
Then
\[
\int_T |f(\varphi(rz))|^p dm(z) \leq \int_T \int_T P_{\varphi(rz)}(\zeta) |f(\zeta)|^p dm(\zeta) dm(z)
\]
\[
= \int_T \int_T |f(\zeta)|^p P_{\varphi(rz)}(\zeta) dm(z) dm(\zeta),
\]
and since \( z \to P_{\varphi(rz)}(\zeta) \) is harmonic in \( r^{-1}D \), the inner integral is
\[
\int_T P_{\varphi(rz)}(\zeta) dm(z) = P_{\varphi(0)}(\zeta) \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|},
\]
which proves the first part of the theorem.

To see the second, assume first that equality holds for some \( 0 < p < \infty \) and all \( f \in H^p \). Let \( \zeta \in T \) be fixed but arbitrary, and let \( f(z) = (\zeta + z)^{2/p} \). Then
\[
2 = \| f \|_p^p = \| f \circ \varphi \|_p^p = \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \int_T |\zeta + \varphi(z)|^2 dm(z)
\]
\[
= \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} (1 + 2 \Re \zeta \varphi(0) + 1).
\]
Clearly, the last expression on the right depends on \( \zeta \) and cannot equal \( 2 \), unless \( \varphi(0) = 0 \). To see that \( \varphi \) is inner, let \( f(z) = z \), to obtain
\[
1 = \| f \|_p^p = \| f \circ \varphi \|_p^p = \int_T |\varphi(\zeta)|^p dm(\zeta),
\]
which immediately \( |\varphi(\zeta)| = 1 \), \( m \)-a.e., because, by the assumption of the theorem we have \( |\varphi(\zeta)| \leq 1 \), \( m \)-a.e..

Conversely, assume that \( \varphi \) is inner with \( \varphi(0) = 0 \), and let \( Q(z) = \sum_n a_n z^n \) be any polynomial. Then
\[
\| Q \circ \varphi \|_2^2 = \sum_{n,k} a_n \overline{a_k} \int_T \varphi^n(\zeta) \overline{\varphi^k(\zeta)} dm(\zeta) = \sum_n |a_n|^2 = \| Q \|_2^2,
\]
since for \( n > k \)
\[
\int_T \varphi^n(\zeta) \overline{\varphi^k(\zeta)} dm(\zeta) = \int_T \varphi^{n-k}(\zeta) dm(\zeta) = 0.
\]
If \( f \in H^2 \) is arbitrary, we approximate \( f \) in the norm of \( H^2 \) by a sequence of polynomials \( (Q_n) \). By the first part of the theorem \( Q_n \circ \varphi \to f \circ \varphi \) in \( H^2 \), which gives that \( \| f \|_2 = \| f \circ \varphi \|_2 \). In particular, for every inner function \( I \) we have
\[
1 = \| I \|_2 = \| I \circ \varphi \|_2,
which implies that \( I \circ \varphi \) is inner as well. If \( 0 < p < \infty \) is arbitrary, and \( f \in H^p \), we write \( f = Bg \) with \( B \) a Blaschke product and \( g^{p/2} \in H^2 \). Then \( B \circ \varphi \) is inner and
\[
\|f \circ \varphi\|_p = \|g^{p/2} \circ \varphi\|_2^{2/p} = \|g^{p/2}\|_2^{2/p} = \|f\|_p,
\]
and the proof is complete. \( \square \)

This result, especially the first part of it has numerous applications. We shall only mention one of these below.

**Corollary 3.3.** Let \( f \) be analytic in \( \mathbb{D} \) with \( \Re f(z) \geq 0, z \in \mathbb{D} \). Then \( f \) is outer and belongs to \( H^p \) for all \( 0 < p < 1 \).

**Proof.** If \( \phi \) is an automorphism of the disc with \( f(0)^{1+\phi(0)} = 1+\varphi \). Indeed, \( g \) maps the disc onto the right half-plane, hence \( g^{-1} \circ f \) maps \( \mathbb{D} \) into itself with \( g^{-1} \circ f(0) = 0 \). We claim that \( g \in H^p, 0 < p < 1 \). In view of Theorem 3.6 it suffices to show that \( z \to \frac{1+z}{1-z} \) belongs to \( H^p, 0 < p < 1 \), which follows by a direct calculation. Thus, another application of Theorem 3.6 gives that \( f \in H^p, 0 < p < 1 \). Note that \( 1/f \) has nonnegative real part as well, so \( 1/f \in H^p, 0 < p < 1 \), hence by Exercise 2 in the previous subsection, \( f \) is outer. \( \square \)

**3.5. The Phragmen-Lindelöf Principle.** The type of problem we want to investigate in this section is illustrated by the following example.

Let \( 0 < a < \pi \), and suppose that \( f \) is analytic in the the angle \( \Gamma = \{ z = re^{it}, r > 0, |t| < a \} \), and continuous in its closure, and assume there exists \( M > 0 \) such that
\[
|f(re^{\pm ia})| \leq M, \quad r \geq 0.
\]

It is not difficult to show by means of examples that such a function \( f \) does not need to be bounded in \( \Gamma \), i.e. the ”naive” form of the maximum principle fails in this domain. Of course one would like to add conditions that are as weak as possible, under which one can conclude that \( |f| \leq M \) in \( \Gamma \).

A more general situation of this type can be described in the unit disc. Consider an analytic function \( f \) in \( \mathbb{D} \) which has the nontangential limit \( f(\zeta) \), for almost every \( \zeta \in \mathbb{T} \). It turns out that the properties of \( f \) are difficult to deduce from the behavior of the boundary function. For example, if \( S \) is a nonconstant singular inner function, then \( f = 1/S \) has unimodular nontangential limits a.e. on \( \mathbb{T} \), yet \( f \) is unbounded, in fact, it doesn’t belong to any \( H^p, p > 0 \). This is easily seen from the inner-outer factorization, since \( 1/S = JF \) implies \( F = 1 \), hence \( SJ = 1 \).
and by the maximum principle, both \( S \) and \( J \) would be constant. It turns out that this type of situation can be addressed with help of the Phragmen-Lindelöf principle which we present here in the most general form:

**Theorem 3.7.** Let \( F \) be an outer function and \( f \) be an analytic function in \( \mathbb{D} \) such that

\[
|f(z)| \leq |F(z)|
\]

for all \( z \in \mathbb{D} \). Then \( f \) has the nontangential limit \( f(\zeta) \), for almost every \( \zeta \in \mathbb{T} \), and if \( f \in L^p(m) \), for some \( p \in (0, \infty] \), then \( f \in H^p \).

**Proof.** By assumption we have that \( f/F \in H^\infty \), so that \( f/F \), and hence \( f \) has nontangential limits a.e. on \( \mathbb{T} \). By Proposition 1.1 \( F \) has nonzero nontangential limits a.e. on \( \mathbb{T} \), hence, so has \( f \). The inner-outer factorization of \( f/F \) gives \( f/F = IG \) with \( I \) inner and \( G \) outer, so that \( f = IGF \), and

\[
|f| = |GF|,
\]

\( m \)-a.e. on \( \mathbb{T} \). Then by Corollary 1.1 \( GF \in H^p \), and since \( |f| \leq |GF| \) in \( \mathbb{D} \), it follows that \( f \in H^p \).

**Corollary 3.4.** Let \( 1 \leq p \leq \infty \). The map which associates to \( f \in H^p \) its boundary function is a linear isometry from \( H^p \) onto the closed subspace of \( L^p(m) \) consisting of functions \( g \) with

\[
\int_{\mathbb{T}} \zeta^n g(\zeta) dm(\zeta) = 0, \quad n \geq 1.
\]

Its inverse is given by the Cauchy formula

\[
f(z) = \int_{\mathbb{T}} \frac{\zeta f(\zeta)}{\zeta - z} dm(\zeta).
\]

**Proof.** It is clear that the boundary functions of \( H^p \)-functions belong to the subspace described in the statement, and also that this subspace is closed. If \( g \in L^p(m) \) with

\[
\int_{\mathbb{T}} \zeta^n g(\zeta) dm(\zeta) = 0, \quad n \geq 1,
\]

then it is the boundary function of an \( H^1 \)-function, and the result follows by Theorem 3.7. The Cauchy formula obviously holds for the dilated function \( f_r, f \in H^p \), and the formula in the statement follows by letting \( r \to 1^- \).

We close with an application of the theorem to the situation described at the beginning of the paragraph.
Theorem 3.8. Suppose that \( f \) is analytic in the angle \( \Gamma = \{ z = re^{it}, r > 0, |t| < a \} \), where \( 0 < a < \pi \) is fixed. Assume there exists \( M > 0 \) such that for every boundary point \( \zeta = re^{\pm ia} \) we have
\[
\limsup_{z \to \zeta, z \in \Gamma} |f(z)| \leq M.
\]
If \( f \) satisfies in addition an inequality of the form
\[
|f(z)| \leq b \exp(c|z|^d)
\]
for some constants \( b, c > 0, 0 < d < \frac{\pi}{2a} \) and all \( z \in \Gamma \) then \( f \) is bounded in the whole angle with \( |f(z)| \leq M, z \in \Gamma \).

Proof. Under our assumption on the constant \( d \), there is \( c_1 > 0 \) such that \( |z|^d \leq \Re z^d \), hence
\[
|f(z)| \leq b \exp(c_2 \Re z^d), \quad z \in \Gamma.
\]
Let \( \psi : \mathbb{D} \to \Gamma \), \( \psi(z) = \frac{1+z}{1-z} e^{-2a/\pi} \). The \( \psi \) maps \( \mathbb{D} \) conformally onto \( \Gamma \), so that it suffices to show that \( f \circ \psi \in H^\infty \) with \( \|f \circ \psi\|_\infty \leq M \). By the first inequality in this proof we have that
\[
|f \circ \psi(z)| \leq |e^{\psi^d(z)}| = |F(z)|.
\]
We claim that \( F \) is outer. If the claim holds, the result follows by Theorem 3.7, since the nontangential limits of \( \psi \) lie on \( \partial \Gamma \) a.e. and by assumption it follows that the nontangential limits of \( |f| \) are \( \leq M \) a.e. To see the claim, recall from the previous subsection that \( \psi^{\pi/2a} \in H^p \), \( 0 < p < 1 \), which implies that \( \Re \psi^d \in h^q \), for some \( q > 1 \), hence it is the Poisson integral of its boundary function, which is obviously integrable. Thus \( F \) is outer and the proof is complete. \( \Box \)
CHAPTER 4

The dual of $H^p$

1. Basic reduction to the study of Cauchy integrals

We shall be concerned with the continuous linear functionals on the Banach spaces $H^p$, $1 \leq p < \infty$, that is the linear maps $l : H^p \to \mathbb{C}$ with

$$|l(f)| \leq C\|f\|_p,$$

for some $C > 0$ and all $f \in H^p$.

The description of these functionals is particularly simple when $p = 2$, since $H^2$ is a Hilbert space with respect to the scalar product in $L^2(m)$

$$\langle f, g \rangle = \int_T f g dm, \quad f, g \in H^2.$$

According to the Riesz representation theorem, every continuous linear functional $l$ on $H^2$ has the form

$$l(f) = \langle f, g \rangle = \int_T f g dm, \quad f \in H^2,$$

for some fixed $g \in H^2$.

When $p \neq 2$, general functional analysis provides less information. However we can use Corollary 3.4 to embed $H^p$ into $L^p(m)$ and then use the Hahn-Banach theorem to represent every continuous linear functional $l$ on $H^p$, $1 \leq p < \infty$ by

$$l(f) = \int_T f g dm,$$

with $g \in L^q(m)$, $\frac{1}{p} + \frac{1}{q} = 1$. The problem that arises here, is that this representation is not unique, it leads to a representation of the dual of $H^p$ as a quotient space. This is a well known fact in functional analysis and follows by another application of the Hahn-Banach theorem. The proof of the proposition below is left as an exercise. As usual we denote the dual of the normed space $X$ by $X'$ and if $S \subset X$,

$$S^\perp = \{l \in X' : l(S) = \{0\}\}.$$
Proposition 1.1. Let $X$ be a Banach space and let $Y$ be a closed subspace. Then the map $J : X'/Y^\perp \rightarrow Y'$,

$$J[l] = l|Y,$$

is a linear isometry.

According to this result, $(H^p)'$ is isometrically isomorphic to the quotient space $L^q(m)/(H^p)^\perp$. Moreover, $(H^p)^\perp$ consists of functions $g \in L^q(m)$ with

$$\int_\mathbb{T} \zeta^n g(\zeta) dm(\zeta) = 0, \quad n \geq 0,$$

hence by the F. & M. Riesz theorem, it follows easily that $(H^p)^\perp = H^q_0$, the subspace of $H^q$ consisting of functions that vanish at the origin.

Can we "single out" elements in the cosets $[f] \in L^q(m)/H^q_0$ in a meaningful way?

There is a simple but powerful idea that can be used here, namely to consider Cauchy integrals. Note that if $g \in (H^p)^\perp$ then

$$\int_\mathbb{T} \frac{\tilde{g}(\zeta)}{\zeta - \tilde{z}} dm(\zeta) = 0, \quad z \in \mathbb{D}$$

since

$$\frac{\tilde{\zeta}}{\zeta - \tilde{z}} = \sum_{n=0}^{\infty} \frac{\tilde{z}^n}{\zeta^n} = \sum_{n=0}^{\infty} \frac{z^n \zeta^n}{\zeta^n}, \quad \zeta \in \mathbb{T}, z \in \mathbb{D},$$

and the series converges uniformly on $\mathbb{T}$ when $z \in \mathbb{D}$ is fixed. This gives the following result which is the starting point for the material of this chapter.

Proposition 1.2. Let $l$ be a continuous linear functional on $H^p$, $1 \leq p < \infty$. Then there exists a unique analytic function $h$ in $\mathbb{D}$ of the form

$$h(z) = \int_\mathbb{T} \frac{\bar{g}(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \mathbb{D},$$

where $g \in L^q(m)$, $\|g\|_{L^q(m)} \leq 2\|l\|$, such that

$$l(f) = \lim_{r \rightarrow 1-} \int_\mathbb{T} f(\zeta) \bar{h}(rz) dm(z) \quad f \in H^p.$$

Proof. We can write

$$l(f) = \int_\mathbb{T} f(\zeta) g(\zeta) dm(\zeta),$$
with \([g] \in L^q(m)/(H^p)_{\perp}\) and \([g]_{L^q(m)} \leq 2\|\|g\|\| = \|l\|\). Moreover, since \(f_r \to f\) in \(H^p\), when \(r \to 1^-\) we obtain easily that \(l(f_r) \to l(f)\), i.e.

\[
l(f) = \lim_{r \to 1^-} \int_{\mathbb{T}} f(r\zeta)g(\zeta)dm(\zeta).
\]

Using the Cauchy formula as in Corollary 3.4 we have for \(\zeta \in \mathbb{T}\)

\[
f(r\zeta) = \int_{\mathbb{T}} \frac{zf(z)}{z - r\zeta} dm(z), \quad \zeta \in \mathbb{T}, 0 < r < 1,
\]

and by an application of Fubini’s theorem we obtain

\[
\int_{\mathbb{T}} f(r\zeta)g(\zeta)dm(\zeta) = \int_{\mathbb{T}} zf(z) \int_{\mathbb{T}} \frac{g(\zeta)}{z - r\zeta} dm(\zeta)dm(z) = \int_{\mathbb{T}} f(z) \int_{\mathbb{T}} \frac{\bar{g}(\zeta)}{\zeta - rz} dm(\zeta)dm(z).
\]

Then the result follows with

\[
h(z) = \int_{\mathbb{T}} \frac{\bar{g}(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \mathbb{D},
\]

using the considerations preceding the proposition. \(\Box\)

According to this result, one way to describe the continuous linear functionals on \(H^p\) is to understand Cauchy integrals of \(L^q(m)\)-functions \(\frac{1}{p} + \frac{1}{q} = 1\), and this is precisely what will be done in the sequel.

The Cauchy integral, or the Cauchy transform of \(h \in L^1(m)\) is denoted by \(\hat{h}\) and is defined by

\[
(1.15) \quad \hat{h}(z) = \int_{\mathbb{T}} \frac{h(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \mathbb{C} \setminus \mathbb{T}.
\]

This formula gives 2 analytic functions, one in \(\mathbb{D}\), and the other in \(\mathbb{C} \setminus \mathbb{D}\), but we will be mostly concerned with the first, since the properties of the second are very similar. In fact their boundary values are related by the so-called "jump theorem".

**Theorem 1.1.** If \(h \in L^1(m)\)

\[
\lim_{r \to 1^-} \hat{h}(rz) - \hat{h}(\frac{1}{r}z) = \bar{z}h(z)
\]

\(m\)-a.e. on \(\mathbb{T}\).

**Proof.** It suffices to check the simple identity

\[
\frac{1}{\zeta - rz} - \frac{1}{\zeta - r^{-1}z} = \bar{\zeta}P_z(\zeta)
\]

and apply the results about boundary behavior of Poisson integrals. \(\Box\)
2. The M. Riesz theorem

In view of the representation as Cauchy integrals of $H^p$-functions, $1 \leq p \leq \infty$, proved in Corollary 3.4, a natural question that comes to mind is whether the Cauchy transforms of $L^p(m)$-functions belong to $H^p$, $1 \leq p \leq \infty$?

For real-valued functions, the above question turns into a question about harmonic conjugation. Indeed, if $h \in L^1(m)$ is real-valued with Poisson integral $u$, then the analytic function

$$f(z) = \int \frac{\zeta + z}{\zeta - z} h(\zeta) d\mu(\zeta), \quad z \in D,$$

satisfies

$$\text{Re} f(z) = u(z), \quad z \in D, \quad f(0) \in \mathbb{R}.$$ 

Thus for such functions, we can reformulate the above in terms of real parts:

If $f$ is analytic in $D$ and $\text{Re} f \in h^p$, $1 \leq p \leq \infty$, does $f \in H^p$ (or $\text{Im} f \in h^p$)?

Obviously the original question reduces to real-valued functions, so that the two are equivalent.

It turns out that the answer is negative for the "endpoints" $p = 1$ and $p = \infty$. Indeed, if

$$f_1(z) = \frac{1 + z}{1 - z}, \quad f_2(z) = i \log(1 - z), \quad z \in D,$$

then $\text{Re} f_1(z) = P_z(1)$, so that $\text{Re} f_1 \in h^1$, and $\text{Re} f_2 \in h^\infty$. However, $f_1 \notin h^1$, because its boundary function is not integrable, and $f_2$ is obviously unbounded.

Nevertheless, the questions discussed above have an affirmative answer for all $1 < p < \infty$. This is the content of a famous theorem proved by M. Riesz.

**Theorem 2.1. (M. Riesz)** For $1 < p < \infty$, we have that if $h \in L^p(m)$ then $\hat{h} \in H^p$, and there exists $C_p > 0$ such that

$$\|\hat{h}\|_p \leq C_p \|h\|_{L^p(m)}.$$

Equivalently, if $f$ is analytic in $D$, $f(0) \in \mathbb{R}$ and $\text{Re} f \in h^p$, then $f \in H^p$ and there exists $C'_p > 0$ such that

$$\|f\|_p \leq C'_p \|\text{Re} f\|_p.$$
2. THE M. RIESZ THEOREM

Proof. When \( 1 < p \leq 2 \) we shall prove the second statement. Let \( u = \text{Re} f, \ v = \text{Im} f \). We use the Green formula in the form (6.12), i.e.
\[
\int_T \varphi dm = \varphi(0) + \int_D \Delta \varphi(z) \log \frac{1}{|z|} dA(z).
\]
to compute for \( 0 < r < 1 \), \( \| f_r \|_p, \| u_r \|_p \). A straightforward computation based on the Cauchy-Riemann equations yields
\[
\Delta |f_r|^p = p^2 |f_r|^{p-2} |f'_r|^2,
\]
\[
\Delta |u_r|^p = p(p-1)u_r^{p-2} \left[ \left( \frac{\partial u_r}{\partial x} \right)^2 + \left( \frac{\partial u_r}{\partial y} \right)^2 \right]
\]
\[= p(p-1)u_r^{p-2} |f'|^2.\]
Thus for \( 1 < p \leq 2 \) we have
\[
\Delta |f_r|^p \leq \frac{p}{p-1} \Delta |u_r|^p,
\]
hence by (6.12) and the fact that \( v(0) = 0 \) we obtain
\[
\| f_r \|_p \leq \frac{p}{p-1} \| u_r \|_p,
\]
and the result follows letting \( r \to 1^- \).

In the remaining case when \( 2 < p < \infty \), we shall prove the first statement by duality. Let \( h \in L^p(m) \), and let \( g \in L^q(m) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). For \( 0 < r < 1 \) we have by Fubini’s theorem,
\[
\int_T \hat{h}(r\zeta)g(\zeta)dm(\zeta) = \int_T h(z) \int_T \frac{g(\zeta)}{z - r\zeta} dm(\zeta) dm(z).
\]
As in the proof of Proposition 1.2 we have for \( z \in T \),
\[
\int_T \frac{g(\zeta)}{z - r\zeta} dm(\zeta) = z \int_T \frac{\bar{g}(\zeta)}{\zeta - rz} dm(\zeta) = \bar{z} \hat{g}_1(z),
\]
where \( g_1(\zeta) = \bar{\zeta}g(\zeta) \). Using the first part of the proof, the fact that \( 1 < q < 2 \), and the equivalence of the two statements, it follows that \( G(z) = z \hat{g}_1(z) \) belongs to \( H^q \) with
\[
\| G \|_q \leq \frac{q}{q-1} \| g \|_{L^q(m)}.
\]
Thus by Hölder’s inequality
\[
\left| \int_T \hat{h}(r\zeta)g(\zeta)dm(\zeta) \right| \leq \| h \|_{L^p(m)} \| G \|_q \leq \left( \frac{q}{q-1} \right)^{1/q} \| h \|_{L^p(m)} \| g \|_{L^q(m)}.
\]
which gives
\[ \|\hat{h}\|_p \leq \left( \frac{q}{q - 1} \right)^{1/q} \|h\|_{L^p(m)}, \]
and the result follows letting \( r \to 1^- \). \( \square \)

Combining the M. Riesz theorem with Proposition 1.2 we obtain immediately a nice description of the dual of \( H^p \) for \( 1 < p < \infty \).

**Corollary 2.1.** For every continuous linear functional \( l \) on \( H^p, 1 < p < \infty \), there exists a unique \( g \in H^q, \frac{1}{p} + \frac{1}{q} = 1 \), such that
\[ l(f) = \int_T f(\zeta)\overline{g(\zeta)}dm(\zeta), \quad f \in H^p. \]
Moreover, there exists a positive constant \( a_p \) such that
\[ a_p \|g\|_q \leq \|l\| \leq \|g\|_q. \]

**Exercise 2.** Prove the M. Riesz theorem in the case \( p = 2 \) using the Parseval formula.

### 3. The dual of \( H^1 \)

According to the discussion at the beginning of this chapter, in order to represent the dual of \( H^1 \) in a useful way, as a space of analytic functions, we need to understand the Cauchy integrals of \( L^\infty(m) \)-functions. As we have seen in the previous section, such functions may fail to be bounded, because the M. Riesz theorem fails for \( p = \infty \). However, a description of Cauchy integrals of \( L^\infty(m) \)-functions can be obtained using the following computation based on conformal invariance.

**Lemma 3.1.** Let \( h \in L^1(m) \) and let \( \hat{h} \) be its Cauchy transform. If \( a \in \mathbb{D}, \) and \( \phi_a(z) = \frac{a - z}{1 - \bar{a}z}, \) \( z \in \mathbb{D}, \) then
\[ \hat{h}(\phi_a(z)) - \hat{h}(a) = z\hat{h}_1(z), \quad z \in \mathbb{D}, \]
where \( h_1(\zeta) = \overline{\phi_a(\zeta)}h(\phi_a(\zeta)), \) \( \zeta \in \mathbb{T}. \)

**Proof.** Start with
\[ \frac{1}{\zeta - \phi_a(z)} - \frac{1}{\zeta - a} = \frac{\phi_a(z) - a}{(\zeta - \phi_a(z))(\zeta - a)}, \]
and
\[ \phi_a(z) - a = -\frac{(1 - |a|^2)z}{1 - \bar{a}z}. \]
Moreover,

\[ \zeta - \phi_a(z) = \frac{\zeta - a\zeta z - a + z}{1 - az} = \frac{1 - a\zeta}{1 - az}(-\phi_a(\zeta) + z). \]

From these three computations we obtain

\[ \frac{1}{\zeta - \phi_a(z)} - \frac{1}{\zeta - a} = \frac{(1 - |a|^2)z}{(\phi_a(\zeta) - z)(1 - a\zeta)(\zeta - a)}, \quad z \in \mathbb{D}, \zeta \in \mathbb{T}, \]

and since \( \zeta \in \mathbb{T} \), we have \( \zeta = \frac{1}{\zeta} \), i.e.

\[ \frac{1}{\zeta - \phi_a(z)} - \frac{1}{\zeta - a} = \frac{z\bar{\zeta}}{(\phi_a(\zeta) - z)(1 - a\zeta)} = \frac{z\bar{\zeta}}{(\phi_a(\zeta) - z)} |\phi_a'(\zeta)|. \]

Thus,

\[ \hat{h}(\phi_a(z)) - \hat{h}(a) = z \int_{\mathbb{T}} \frac{\bar{\zeta}}{(\phi_a(\zeta) - z)} h(\zeta) |\phi_a'(\zeta)| dm(\zeta). \]

Now use the fact that \( \phi_a^{-1} = \phi_a \), together with the change of variable

\[ \int_{\mathbb{T}} g \circ \phi_a |\phi_a'| dm = \int_{\mathbb{T}} g dm, \]

to obtain

\[ \hat{h}(\phi_a(z)) - \hat{h}(a) = z \int_{\mathbb{T}} \frac{\phi_a(\zeta) h(\phi_a(\zeta))}{\zeta - z} dm(\zeta), \]

which is the formula in the statement. \( \square \)

This useful result reveals a remarkable property of Cauchy transforms of \( L^\infty \)-functions. Note first that by the M. Riesz theorem, these Cauchy transforms belong to \( H^p \), for all \( p > 0 \).

**Proposition 3.1.** There exists \( C > 0 \), such that if \( h \in L^\infty(m) \), and \( \phi_a(z) = \frac{a - z}{1 - az} \), \( a, z \in \mathbb{D} \), then

\[ \sup_{a \in \mathbb{D}} \| \hat{h} - \hat{h}(a) \|_2 \leq C \| h \|_{L^\infty(m)}. \]

In other words, the boundary function of \( \hat{h} \) belongs to \( BMO \).

**Proof.** Apply the M. Riesz theorem in the case \( p = 2 \), together with Lemma 3.1 to obtain

\[ \| \hat{h} - \hat{h}(a) \|_2 \leq C \| \overline{\phi_a h} \circ \phi_a \|_{L^2(m)} \leq C \| h \|_{L^\infty(m)}. \] \( \square \)
Let us denote by \( BMOA \) the space of analytic functions in \( BMO \), more precisely, the space of functions in \( H^2 \), whose boundary valued belong to \( BMO \). The more striking result is that Cauchy transforms of \( L^\infty(m) \)-functions actually coincide with \( BMOA \).

**Theorem 3.1.** A function belongs to \( BMOA \) if and only if it is the Cauchy transform of an element of \( L^\infty(m) \).

**Proof.** The fact that the Cauchy transform of an element of \( L^\infty(m) \) belongs to \( BMOA \) has already been established in the previous proposition. To see the converse, we use Proposition 1.2 from the beginning of this chapter. According to this result, a function \( g \in BMOA \) is the Cauchy transform of an \( L^\infty(m) \)-function if the limit

\[
 l(f) = \lim_{r \to 1^-} \int_T f \overline{g} dm,
\]

exists and defines a bounded linear functional on \( H^1 \). Observe that since \( g \in H^2 \) we can apply Parseval’s formula to obtain

\[
 \int_T f \overline{g} dm = \int_T f_r \overline{g} dm, \quad f \in H^1, \quad 0 < r < 1.
\]

Finally, let us note that it suffices to show an estimate of the form

\[
 (3.16) \quad \left| \int_T f \overline{g} dm \right| \leq C \| f \|_1,
\]

for some \( C > 0 \) and all \( f \in H^1, 0 < r < 1 \). Indeed, if (3.16) holds then for \( r, \rho \to 1^- \), \( r > \rho \), we have

\[
 f_r - f_\rho = \left( f - f_\rho/r \right)_r,
\]

hence

\[
 \left| \int_T (f_r - f_\rho) \overline{g} dm \right| \leq C \| f - f_\rho/r \|_1 \to 0,
\]

i.e. the limit \( l(f) \) exists.

We proceed in three steps. First we use the Littlewood-Paley formula Proposition 6.2 in Chapter 2 in polarized form to obtain

\[
 \int_T f_r \overline{g} dm = f(0) \overline{g}(0) + \int_T (f_r - f(0)) g - \overline{g}(0) dm
\]

\[
 = f(0) \overline{g}(0) + \int_D \nabla f_r \cdot \nabla \log \frac{1}{|z|} dA
\]

\[
 = f(0) \overline{g}(0) + 2 \int_D f'_r g' \log \frac{1}{|z|} dA.
\]
Next we write \( f \in H^1 \) as a product \( f = uv \), with \( u, v \in H^2 \), \( \|u\|_2^2 = \|v\|_2^2 = \|f\|_1 \). This follows easily if we write \( f = Bh \), with \( B \) a Blaschke product and \( h \in H^1 \), zero-free in \( D \), and set \( u = Bh^{1/2} \), \( v = h^{1/2} \). Then

\[
\int_T f \overline{f} \, dm = f(0) \overline{g}(0) + 2 \int_D u'_r v_r g' \log \frac{1}{|z|} \, dA + 2 \int_D u'_r v_r g' \log \frac{1}{|z|} \, dA.
\]

Obviously, it suffices to estimate the first integral on the right hand side, since the second is completely analogous. By the Cauchy-Schwarz inequality

\[
\left| \int_D u'_r v_r g' \log \frac{1}{|z|} \, dA \right|^2 \leq \int_D |u'_r|^2 \log \frac{1}{|z|} \, dA \int_D |v_r|^2 |g'|^2 \log \frac{1}{|z|} \, dA.
\]

Another application of the littlewood-Paley formula gives

\[
\int_D |u'_r|^2 \log \frac{1}{|z|} \, dA \leq \frac{1}{2} \|u_r\|_2^2 \leq \frac{1}{2} \|u\|_2^2 = \|f\|_1.
\]

To see the last integral, apply Corollary 6.1 in Chapter 2 to conclude that if \( g \in BMOA \), the measure

\[
|\nabla g(z)|^2 \log \frac{1}{|z|} \, dA(z) = 2|g'(z)|^2 \log \frac{1}{|z|} \, dA(z),
\]

is a Carleson measure, so that

\[
\int_D |v_r|^2 |g'|^2 \log \frac{1}{|z|} \, dA \leq C\|v_r\|_2^2 \leq C\|v\|_2^2 = \|f\|_1.
\]

The proof is complete. \( \square \)

From the discussion at the beginning of the chapter we now obtain the following description of the dual of \( H^1 \). Until now, for the definition of \( BMOA \) we have used the seminorm

\[
sup_{a \in \partial} \|g \circ \phi_a - g(a)\|_2.
\]

A natural norm on this space is

\[
\|g\|_* = |g(0)| + \sup_{a \in \partial} \|g \circ \phi_a - g(a)\|_2.
\]

**Corollary 3.1.** A linear functional \( l : H^1 \rightarrow \mathbb{C} \) is continuous if and only if it has the form

\[
l(f) = \lim_{r \rightarrow 1^-} \int_T f \overline{g}_r \, dm,
\]

for some function \( g \in BMOA \). Moreover, \( g \) is uniquely determined by \( l \) and there exist absolute constants \( a, b > 0 \) such that

\[
a\|l\| \leq \|g\|_* \leq b\|l\|.
\]
4. The John-Nirenberg inequality

BMOA is a very interesting space of functions with a number of surprising properties. Let us note from the beginning that the study of the analytic functions with bounded mean oscillation, is not too restrictive, since many of their properties continue to hold for all such functions. The reason is provided below.

**Proposition 4.1.** The function \( f \) belongs to BMOA if and only if the boundary function of its real part belongs to BMO.

**Proof.** Let \( u = \text{Re} f \). Since \( |\nabla u|^2 = 2|f'|^2 \), the measure \( |\nabla u|^2 \log \frac{1}{|z|} dA \) is Carleson if and only if \( |f'|^2 \log \frac{1}{|z|} dA \) is. \( \square \)

**Exercise 1.** Show that every function in BMOA is a linear combination of four functions in BMOA whose real part is bounded.

Recall that by the M. Riesz theorem it follows that BMOA \( \subset \bigcap_{p>0} H^p \), but it is not contained in \( H^\infty \). Since any real-valued function in BMO equals the boundary values of the real part of a BMOA function if follows also that BMO \( \subset \bigcap_{p>0} L^p(m) \). The estimate below is much stronger than that and has far-reaching consequences.

**Theorem 4.1.** Let \( g \in \text{BMOA} \). Then there exists \( c > 0 \) such that \( e^g \in H^2 \). If \( h \in \text{BMO} \) then there exists \( d > 0 \) such that \( e^{d|h|} \in L^2(m) \).

**Proof.** The slick argument below is due to Pommerenke. Given \( g \in \text{BMOA} \), and assume without loss of generality that \( g(0) = 0 \). Consider the linear map defined on \( H^2 \) by

\[
T_g f(z) = \int_0^z f(\zeta)g'(\zeta) d\zeta, \quad z \in \mathbb{D}, \quad f \in H^2.
\]

We claim that \( T_g \) is a bounded linear operator from \( H^2 \) into itself. Indeed, by the Littlewood-Paley formula and the fact that \( g \in \text{BMOA} \) we obtain

\[
\|T_g f\|_2^2 = 2 \int_D |(T_g f)'|^2 \log \frac{1}{|z|} dA = 2 \int_D |fg'|^2 \log \frac{1}{|z|} dA \leq C\|f\|_2^2,
\]

and the claim follows. Then by elementary functional analysis it follows that for \( |\lambda| > \|T_g\| \), \( \lambda I - T_g \) is invertible on \( H^2 \), i.e., for every \( F \in H^2 \) there is a unique solution \( f \in H^2 \) of

\[
\lambda f - T_g f = F.
\]
A simple calculation shows that for $F = 1$, the solution $f$ is given by

$$f = e^{g/\lambda},$$

and if we choose $\lambda > 0$, the first part follows. Also note that

$$e^{-g/\lambda}, e^{\pm ig/\lambda} \in H^2,$$

as well. This gives that $e^{\pm Re g/\lambda}, e^{\pm Im g/\lambda} \in L^2(m)$, and together with the above discussion, this immediately implies the second part. $\square$
Sequences and families of holomorphic functions

One of the basic applications of the Cauchy formula is the fact that the set of holomorphic functions in a given region in the complex plane is closed with respect to uniform convergence on compacts. That is, given any sequence \((f_n)\) of holomorphic functions in the open set \(U \subset \mathbb{C}\) which converges uniformly on every compact subset of \(U\) to a function \(f : U \to \mathbb{C}\), it follows that \(f\) is also holomorphic on \(U\). This is an important result which, in particular, enables us to construct highly nontrivial examples of holomorphic functions. Some of these are listed below as exercises.

**Exercise 1.** Show that the following expressions define holomorphic functions in the specified domain:

1. \(\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{Re} \ z > 1, \quad \text{(Zeta function)}\)
2. \(\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \text{Re} \ z > 0, \quad \text{(Gamma function)}\)
3. \(f(z) = \sum_{n=1}^{\infty} \frac{1}{(z-n)^2}, \quad z \in \mathbb{C} \setminus \mathbb{N}, \quad \text{(no special name)}\)
4. \(\wp(z) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}, m^2 + n^2 \neq 0}^{\infty} \left[ \frac{1}{(z-m-in)^2} - \frac{1}{(m+in)^2} \right], \quad z \in \mathbb{C} \setminus \mathbb{Z} + i\mathbb{Z}, \quad \text{(Weierstrass function)}\)

**Theorem 0.2.** Let \((f_n)\) be a sequence of holomorphic functions on the open connected set \(U \subset \mathbb{C}\) that converges uniformly on compacts to the (holomorphic) function \(f\). Let \(V\) be an open subset of \(U\) such that there
exists \( n_0 \geq 1 \) with \( f_n(z) \neq 0 \) for all \( z \in V \) and \( n \geq n_0 \). Then either \( f \) is the constant function 0, or \( f \) has no zeros in \( V \).

**Proof.** Suppose \( f \) is not identically zero, but has a zero \( z_0 \in V \). Choose a small disc \( \Delta \) centered at \( z_0 \) such that \( \Delta \subset V \) and that \( f \) has no zero on \( \partial \Delta \). Since for \( n \geq n_0 \) \( f_n \) has no zero in \( \Delta \) we have
\[
\int_{\partial \Delta} \frac{f'_n(z)dz}{f_n(z)} = 0, \quad n \geq n_0.
\]
On the other hand, \( \frac{f'_n}{f_n} \) converges uniformly on \( \partial \Delta \) to \( \frac{f'}{f} \), so that
\[
\lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f'_n(z)dz}{f_n(z)} = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f'(z)dz}{f(z)} \geq 1
\]
which gives a contradiction and the theorem is proved. \( \square \)

Let us now turn our attention to an important property of families of holomorphic functions that resembles to (relative) compactness of sets in \( \mathbb{R}^n \). We begin with the following general definition.

**Definition 0.1.** A family \( \mathcal{F} \) of holomorphic functions on the open set \( U \subset \mathbb{C} \) is called normal if every sequence \((f_n)\) in \( \mathcal{F} \) contains a subsequence \((f_{n_k})\) that converges uniformly on compact subsets to some (holomorphic) function \( f \) (not necessarily in \( \mathcal{F} \!\)).

The following classical result due to Montel characterizes the normal families of holomorphic functions.

**Theorem 0.3.** (Montel). A family \( \mathcal{F} \) of holomorphic functions on the open set \( U \subset \mathbb{C} \) is normal if it is uniformly bounded on compacts, that is, for every compact set \( K \subset U \) there exists a positive constant \( C_K \) such that for all \( f \in \mathcal{F} \) and all \( z \in K \) we have
\[
|f(z)| \leq C_K.
\]

**Proof.** The result will follow by a standard diagonalization process, once we show that the family \( \mathcal{F} \) is equicontinuous at every point \( z_0 \in U \), that is, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every \( z \in U \) with \( |z - z_0| < \delta \) and all \( f \in \mathcal{F} \) we have
\[
|f(z) - f(z_0)| < \varepsilon.
\]
Assume this claim for a moment and proceed as follows. Choose a countable dense set \( \{z_n, \ n = 1, 2, \ldots\} \) in \( U \) (i.e. for every \( z \in U \) and every neighborhood \( V \) of \( z \) there exists \( n \) such that \( z_n \in V \)). Since the sequences \((f_n(z_j))_{n \geq 1}\) are bounded in \( \mathbb{C} \) we can extract a subsequence
(fnk) such that (fnk1(zl)) converges. This sequence has at its turn a subsequence (fnk2) such that (fnk2(zl2)) converges. If we continue this process indefinitely we obtain subsequences (fnkl) such that (fnkl(zl)) converges for 1 \leq l \leq j. If we set nk = nkl then clearly, the subsequence (f(k)) has the property that (f(k)(zj)) converges for all j = 1, 2, …. This is called a diagonalization process.

Let us now show that (fnk) converges uniformly on compacts. Note first that if w \in U and \varepsilon > 0 then by the claim (equicontinuity) there is \delta > 0 such that

|fnk(z) - fnk(w)| < \varepsilon

for all k \geq 1 and z \in U with |z - w| < \delta. Then, if z, \zeta \in U satisfy |z - w| < \delta and |\zeta - w| < \delta, we have

|fnk(z) - fnk(\zeta)| \leq |fnk(z) - fnk(w)| + |fnk(\zeta) - fnk(w)| < 2\varepsilon.

Fix a point zj (j depends on w) in the above set, such that |zj - w| < \delta. Then there exists k0 depending only on w such that for k, p \geq k0

|fnk(zj) - fnp(zj)| < \varepsilon.

Putting this together we obtain for all z \in U with |z - w| < \delta that

|fnk(z) - fnp(z)| \leq |fnk(z) - fnk(zj)| + |fnk(zj) - fnp(zj)| + |fnp(zj) - fnp(z)| < 3\varepsilon

for all k, p \geq k0.

Finally, let K \subset U be compact \varepsilon, \delta > 0 as above and cover K by finitely many sets of the form \{z, |z - w| < \delta\} \cap U, w \in K. Then the above reasoning shows that there exists k1 \geq 1 such that for all z \in K and k, p \geq k1 we have

|fnk(z) - fnp(z)| < 3\varepsilon.

Since K was arbitrary we obtain that (fnk) converges pointwise to some function f and from the above inequality, letting p \rightarrow \infty we get that for every compact set K \subset U there exists k1 \geq 1 such that for all z \in K and k \geq k1 we have

|fnk(z) - f(z)| < 3\varepsilon,

i.e. (fnk) converges uniformly on compacts to f.

It remains to prove the claim from the beginning of the proof, i.e. the equicontinuity of the family F. For z0 \in U let \Delta_0 be a disc of radius r0 > 0 with \overline{\Delta_0} \subset U. By hypothesis, we know that there exists a constant C0 such that

\sup_{z \in \Delta_0} |f(z)| \leq C0
for all $f \in \mathcal{F}$. By Cauchy’s formula we have
\[
f(z) - f(z_0) = \frac{1}{2\pi i} \int_{\partial \Delta_0} f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) d\zeta = \frac{z - z_0}{2\pi i} \int_{\partial \Delta_0} f(\zeta) \frac{d\zeta}{(\zeta - z)(\zeta - z_0)}
\]
for all $f \in \mathcal{F}$ and all $z \in \Delta_0$. This leads to the estimate
\[
|f(z) - f(z_0)| \leq |z - z_0| \sup_{|\zeta - z_0| = r_0} |f(\zeta)| \frac{1}{|\zeta - z|} \leq C_0 |z - z_0| \frac{1}{r_0 - |z - z_0|}.
\]
Thus, if $\varepsilon > 0$ is given, choose $0 < \delta < r_0$ such that $C_0 \delta/(r_0 - \delta) < \varepsilon$. Then for $|z - z_0| < \delta$ and $f \in \mathcal{F}$ we have
\[
|f(z) - f(z_0)| \leq C_0 \delta \frac{1}{r_0 - \delta} < \varepsilon.
\]
The claim now follows and the proof is complete. \(\Box\)

This is a famous theorem with a large number of applications in complex analysis. One of these applications is a very useful criterion for uniform convergence on compacts.

**Theorem 0.4. (Vitali)** Let $(f_n)$ be a sequence of holomorphic functions on the open connected set $U \subset \mathbb{C}$. Suppose that $\{f_n, n = 1, 2, \ldots\}$ is uniformly bounded on compacts and also that there is a nondiscrete subset $A$ of $U$ such that $(f_n)$ converges pointwise on $A$. Then $(f_n)$ converges uniformly on compacts.

**Proof.** By Montel’s theorem there is at least a subsequence $(f_n_k)$ that converges uniformly on compacts to a holomorphic function $f$ on $U$. Suppose the sequence itself does not converge uniformly on compacts to $f$. Then there exists a compact subset $K$ of $U$, an $\varepsilon > 0$ and a subsequence $(f_{n_k})$ of $(f_n)$ such that
\[
(0.17) \quad \sup_{z \in K} |f_{n_k}(z) - f(z)| \geq \varepsilon
\]
But, again by Montel’s theorem, $(f_{n_k})$ contains at its turn a subsequence $(f_{n_k'})$ that converges uniformly on compacts to some holomorphic function $g$ on $U$. Then letting $p \to \infty$ in (1.1) we see that $g \neq f$. On the other hand, if $A$ is the set in the statement we have, by hypothesis, for all $z \in A$
\[
g(z) = \lim_{p \to \infty} f_{n_k'}(z) = \lim_{n \to \infty} f_n(z) = \lim_{k \to \infty} f_{n_k}(z) = f(z).
\]
Since $A$ is not discrete we obtain $g = f$ by the identity theorem and we arrive at a contradiction that proves our result. \(\Box\)