LUNDS TEKNISKA HÖGSKOLA 2007
MATEMATIK

Formelsamling i Kontinuerliga system.

(Formelsamligen utgör bara ett stöd för minnet. Beteckningar förklaras sålunda ej. Ej heller anges förutsättningar för formlernas giltighet).

Fysikaliska modeller

Kontinuitetsekvationen

$$\frac{\partial q}{\partial t} + \nabla \cdot \boldsymbol{j} = k$$

Diffusion

$$\boldsymbol{j} = -D\nabla u$$

$$\frac{\partial u}{\partial t} - D\Delta u = k$$

$$\text{(Allmännare: } \frac{\partial u}{\partial t} - \nabla \cdot (D\nabla u) = k).$$

Värmeledning

$$\boldsymbol{j} = -\lambda\nabla u, \quad dq = \rho\, c\, du$$

$$\frac{\partial u}{\partial t} - a\Delta u = \frac{a}{\lambda}k \quad \text{där} \quad a = \frac{\lambda}{\rho c}$$

$$\text{(Allmännare: } \rho c\frac{\partial u}{\partial t} - \nabla \cdot (\lambda\nabla u) = k).$$

Elektrostatisk potential

$$\Delta u = -\frac{\rho}{\epsilon\epsilon_0}$$

Svängande sträng och membran

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = \frac{f}{\rho} \quad \text{där} \quad c^2 = \frac{S}{\rho}$$

$$\text{(Allmännare: } \rho\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (S\nabla u) = f).$$

Longitudinella svängningar

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{f}{\rho_l} \quad \text{där} \quad c^2 = \frac{\alpha}{\rho_l}, \quad S = \alpha\frac{\partial u}{\partial x}$$

Svängningar i gaser (ljud)

$$u = \frac{p - p_0}{p_0} \quad \text{(tryckstörning)}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0 \quad \text{där} \quad c^2 = \frac{\gamma p_0}{\rho_0}$$

För svängningar i gaser (ljud) gäller efter linearisering att

$$\begin{cases} \dfrac{1}{\gamma}\dfrac{\partial \tilde{p}}{\partial t} & + & v_0\dfrac{\partial \tilde{v}}{\partial x} & = & 0 \\ v_0\dfrac{\partial \tilde{v}}{\partial t} & + & \dfrac{p_0}{\rho_0}\dfrac{\partial \tilde{p}}{\partial x} & = & 0 \\ & & \tilde{p} & = & \gamma\tilde{\rho} \end{cases}$$

där $\tilde{p} = \dfrac{p - p_0}{p_0}$ och $\tilde{v} = \dfrac{v}{v_0}$.

Vektoranalys

Gauss formel $\displaystyle\int_\Omega \nabla \cdot \boldsymbol{u}\, dV = \oint_{\partial\Omega} \boldsymbol{u} \cdot d\boldsymbol{S}$

Stokes formel $\displaystyle\int_S \nabla \times \boldsymbol{u} \cdot d\boldsymbol{S} = \oint_{\partial S} \boldsymbol{u} \cdot d\boldsymbol{r}$

Greens formel I $\displaystyle\int_\Omega \nabla u \cdot \nabla v\, dV = \oint_{\partial\Omega} u\frac{\partial v}{\partial \boldsymbol{n}}\, dS - \int_\Omega u\Delta v\, dV$

Greens formel II $\displaystyle\int_\Omega (u\Delta v - v\Delta u)\, dV = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) dS$

Laplaceoperatorn i cylindriska koordinater:

$$\begin{aligned} \Delta &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} = \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

Laplaceoperatorn i sfäriska koordinater:

$$\begin{aligned} \Delta &= \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\Lambda = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda = \\ &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda \\ \Lambda &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ \Lambda &= \frac{\partial}{\partial s}(1 - s^2)\frac{\partial}{\partial s} + \frac{1}{1 - s^2}\frac{\partial^2}{\partial\phi^2} \quad \text{om} \quad s = \cos\theta \end{aligned}$$

(θ polardistans, $0 < \theta < \pi$; ϕ longitud, $0 \leq \phi < 2\pi$).

Ortogonalutvecklingar

$$(u \mid v)_w = \int \overline{u(x)} v(x) w(x)\, dx$$

Om $(\varphi_j \mid \varphi_k) = 0,\ j \neq k$, så

$$\begin{cases} u = \sum c_k(u)\varphi_k \\ c_k(u) = \dfrac{(\varphi_k \mid u)}{\rho_k}, \quad \rho_k = (\varphi_k \mid \varphi_k) \end{cases}$$

Parseval:

$$(u \mid v) = \sum \frac{1}{\rho_k} \overline{(\varphi_k \mid u)} (\varphi_k \mid v) = \sum \rho_k \overline{c_k(u)} c_k(v)$$

Sturm-Liouville

$$\mathcal{A}u = \frac{1}{w}(-\nabla \cdot (p\nabla u) + qu)$$

Speciella funktioner

Gammafunktionen och Betafunktionen

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad \Gamma(z+1) = z\Gamma(z), \qquad \Gamma(n+1) = n!, \qquad \Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

$$\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$

Besselfunktioner

$$\begin{aligned} e^{ir\sin\theta} &= \sum_{-\infty}^{\infty} J_n(r) e^{in\theta} \\ J_n(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(z\sin\theta - n\theta)} d\theta, \quad n \quad \text{heltal} \\ J_\nu(z) &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(-\frac{z^2}{4}\right)^k, \quad \nu \neq -1, -2, ... \end{aligned}$$

Bessels differentialekvation

$$u'' + \frac{1}{r}u' + \left(\lambda - \frac{\nu^2}{r^2}\right) u = 0$$

har den allmänna lösningen

$$\begin{cases} aJ_\nu(\sqrt{\lambda}r) + bY_\nu(\sqrt{\lambda}r) & \text{om} \quad \lambda > 0 \\ ar^\nu + br^{-\nu} & \text{om} \quad \lambda = 0,\ \nu \neq 0 \\ a + b\ln r & \text{om} \quad \lambda = \nu = 0\,. \end{cases}$$

Normuttryck

$$\int_0^R \left| J_\nu \left(\frac{r}{R} \alpha_{\nu k} \right) \right|^2 r dr = \frac{R^2}{2} (J_{\nu+1}(\alpha_{\nu k}))^2 = \frac{R^2}{2} (J'_\nu(\alpha_{\nu k}))^2.$$

Nollställen till Besselfunktioner $J_n(x)$

$J_n(\alpha_{nk}) = 0$

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10
1	2.405	3.832	5.136	6.380	7.588	8.772	9.936	11.086	12.225	13.354	14.476
2	5.520	7.016	8.417	9.761	11.065	12.339	13.589	14.821	16.038	17.241	18.433
3	8.654	10.173	11.620	13.015	14.373	15.700	17.004	18.288	19.554	20.807	22.047
4	11.792	13.324	14.796	16.223	17.616	18.980	20.321	21.642	22.945	24.234	25.509
5	14.931	16.471	17.960	19.409	20.827	22.218	23.586	24.935	26.267	27.584	28.887
6	18.071	19.616	21.117	22.583	24.019	25.430	26.820	28.191	29.546	30.885	32.212
7	21.212	22.760	24.270	25.748	27.199	28.627	30.034	31.423	32.796	34.154	35.500
8	24.352	25.904	27.420	28.908	30.371	31.812	33.233	34.637	36.026	37.400	38.762
9	27.494	29.047	30.569	32.065	33.537	34.980	36.422	37.839	39.240	40.628	42.004
10	30.635	32.190	33.716	35.219	36.699	38.160	39.603	41.031	42.444	43.844	45.232

Nollställen till $J'_n(x),\ x < 25$

$J'_n(\alpha'_{nk}) = 0$

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10
1	0.000	1.841	3.054	4.201	5.318	6.416	7.501	8.578	9.647	10.711	11.771
2	3.832	5.331	6.706	8.015	9.282	10.520	11.735	12.932	14.116	15.287	16.448
3	7.016	8.536	9.970	11.346	12.682	13.987	15.262	16.529	17.774	19.005	20.223
4	10.173	11.706	13.170	14.586	15.964	17.313	18.637	19.942	21.229	22.501	23.761
5	13.324	14.864	16.348	17.789	19.196	20.576	21.932	23.268	24.587		
6	16.471	18.016	19.513	20.972	22.401	23.803					
7	19.616	21.164	22.672	24.147							
8	22.760	24.311									
9	25.904										

Sfäriska Besselfunktioner

Differentialekvationen

$$u'' + \frac{2}{z}u' + \left(\lambda - \frac{\ell(\ell+1)}{z^2}\right)u = 0$$

har den allmänna lösningen

$$\begin{cases} a j_\ell(\sqrt{\lambda} z) + b y_\ell(\sqrt{\lambda} z) & \text{om} \quad \lambda > 0 \\ a z^\ell + b z^{-\ell-1} & \text{om} \quad \lambda = 0,\ \ell \neq -\dfrac{1}{2} \\ \dfrac{a + b\ln z}{\sqrt{z}} & \text{om} \quad \lambda = 0, \ell = -\dfrac{1}{2}\,. \end{cases}$$

där

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+1/2}(z), \quad y_\ell(z) = \sqrt{\frac{\pi}{2z}} Y_{\ell+1/2}(z)\ .$$

Speciellt är

$$j_0(z) = \frac{\sin z}{z}\ , \qquad j_1(z) = \frac{\sin z - z\cos z}{z^2}$$

$$y_0(z) = -\frac{\cos z}{z}\ , \quad y_1(z) = -\frac{\cos z + z\sin z}{z^2}$$

Legendrefunktioner

Legendrepolynom $(P_\ell)_0^\infty$ är ortogonala i $L_2(I),\ I = (-1, 1)$.
Legendres differentialekvation

$$\frac{d}{dx}\left((1 - x^2)\frac{du}{dx}\right) + \ell(\ell+1)u = 0, \quad \ell = 0, 1, 2, ...$$

har allmänna lösningen

$$aP_\ell(x) + bQ_\ell(x)$$

där Q_ℓ ej är begränsad i $(-1, 1)$ och

$$P_\ell(x) = \frac{1}{2^\ell \ell!} D^\ell (x^2 - 1)^\ell$$

Rekursionsformel för Legendrepolynom:

$$P_0(x) = 1\ , \quad P_1(x) = x\ , \quad P_{\ell+1}(x) = \frac{2\ell+1}{\ell+1} x P_\ell(x) - \frac{\ell}{\ell+1} P_{\ell-1}(x);$$

Associerade Legrendreekvationen

$$\frac{d}{dx}\left((1 - x^2)\frac{du}{dx}\right) - \frac{m^2}{1 - x^2}u + \ell(\ell+1)u = 0$$

har allmänna lösningen

$$aP_\ell^m(x) + bQ_\ell^m(x)$$

där Q_ℓ^m ej är begränsad och

$$P_\ell^m = (1 - x^2)^{m/2} D^m P_\ell(x)$$

Greenfunktioner

Fundamentallösningar till Laplaceoperatorn ($-\Delta K = \delta$)

$$K(\boldsymbol{x}) = -\frac{1}{2\pi}\ln|\boldsymbol{x}| \quad i \ \ \mathbb{R}^2$$

$$K(\boldsymbol{x}) = \frac{1}{4\pi|\boldsymbol{x}|} \qquad i \ \ \mathbb{R}^3$$

Poissonkärnor

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos\theta} \qquad \text{(enhetscirkeln)}$$

$$P(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad \text{(halvplanet} \quad y>0)$$

Greenfunktion för Dirichlets problem

$$\begin{cases} -\Delta_{\boldsymbol{x}} G(\boldsymbol{x};\boldsymbol{\alpha}) = \delta_{\boldsymbol{\alpha}}(\boldsymbol{x}), & \boldsymbol{x}\in\Omega \\ G(\boldsymbol{x};\boldsymbol{\alpha}) = 0, & \boldsymbol{x}\in\partial\Omega \end{cases}$$

Om $-\Delta u = f$ i Ω, $u = g$ på $\partial\Omega$ så

$$u(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x};\boldsymbol{\alpha}) f(\boldsymbol{\alpha})\,dV_{\boldsymbol{\alpha}} - \oint_{\partial\Omega}\frac{\partial G}{\partial \boldsymbol{n}_{\boldsymbol{\alpha}}}(\boldsymbol{x};\boldsymbol{\alpha})g(\boldsymbol{\alpha})\,dS_{\boldsymbol{\alpha}}$$

Konjugerade punkter med avseende på cirkeln (sfären) $|\boldsymbol{x}| = \rho$

$$|\boldsymbol{\alpha}||\tilde{\boldsymbol{\alpha}}| = \rho^2$$

$$|\boldsymbol{x}-\boldsymbol{\alpha}| = \frac{|\boldsymbol{\alpha}|}{\rho}|\boldsymbol{x}-\tilde{\boldsymbol{\alpha}}| \quad \text{då} \quad |\boldsymbol{x}| = \rho$$

Värmeledning

$$\begin{cases} G(x,t) = \dfrac{1}{\sqrt{4\pi a t}} e^{-x^2/4at} \\ \dfrac{\partial G}{\partial t} - a\dfrac{\partial^2 G}{\partial x^2} = 0, \ x\in\mathbb{R}, \ t>0 \\ G(x,0) = \delta(x), \ x\in\mathbb{R} \end{cases}$$

Vågutbredning

d'Alembert

$$\begin{cases} u(x,t) = \dfrac{1}{2}(g(x-ct)+g(x+ct)) + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} h(y)\,dy \\ g(x) = u(x,0), \ h(x) = u_t(x,0) \end{cases}$$

Karakteristikor

$$\begin{cases} a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x,y,u,u_x,u_y) = 0 \\ a_{11}dy^2 - 2a_{12}dxdy + a_{22}dx^2 = 0 \end{cases}$$

Fouriertransformer

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\,dt$$

$$(\mathcal{F}^{-1}\hat{f})(t) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\hat{f}(\omega)\,d\omega$$

Parsevals formel:

$$\int_{-\infty}^{+\infty} \overline{f(t)}g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{\hat{f}(\omega)}\hat{g}(\omega)\,d\omega$$

$$\xrightarrow{\mathcal{F}}$$

(1)	$\lambda f(t) + \mu g(t)$	$\lambda\hat{f}(\omega) + \mu\hat{g}(\omega)$
(2)	$f(at)$	$\frac{1}{\vert a\vert}\hat{f}\left(\frac{\omega}{a}\right)$
(3)	$f(t-t_0)$	$e^{-it_0\omega}\hat{f}(\omega)$
(4)	$e^{i\omega_0 t}f(t)$	$\hat{f}(\omega-\omega_0)$
(5)	$f'(t)$	$i\omega\hat{f}(\omega)$
(6)	$tf(t)$	$i\frac{d}{d\omega}\hat{f}(\omega)$
(7)	$f * g(t)$	$\hat{f}(\omega)\hat{g}(\omega)$
(8)	$\delta(t)$	1
(9)	1	$2\pi\delta(\omega)$
(10)	$e^{-t}\theta(t)$	$\frac{1}{1+i\omega}$
(11)	$e^{-\vert t\vert}$	$\frac{2}{1+\omega^2}$
(12)	$\frac{1}{1+t^2}$	$\pi e^{-\vert\omega\vert}$
(13)	e^{-t^2}	$\sqrt{\pi}e^{-\omega^2/4}$
(14)	$\theta(t+1)-\theta(t-1)$	$2\frac{\sin\omega}{\omega}$
(15)	$\theta(t)$	$\frac{1}{i}\mathcal{P}\frac{1}{\omega} + \pi\delta(\omega)$

$$\theta(t) = \begin{cases} 1\,, & t > 0 \\ 0\,, & t < 0 \end{cases}$$

Laplacetransformer

$$\begin{aligned}
\mathcal{L}f(s) &= \mathcal{L}_{II}f(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt, \quad \alpha < \operatorname{Re} s < \beta,\ s = \sigma + i\omega \\
f(t) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} F(s)\,ds,\ \alpha < \sigma < \beta \\
\mathcal{F}f(\omega) &= \mathcal{L}_{II}f(i\omega) \\
\mathcal{L}_I f &= \mathcal{L}_{II}(\theta f)
\end{aligned}$$

		$\xrightarrow{\mathcal{L}_{II}}$
(16)	$\lambda f(t) + \mu g(t)$	$\lambda F(s) + \mu G(s)$
(17)	$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{s}{a}\right)$
(18)	$f(t - t_0)$	$e^{-t_0 s} F(s)$
(19)	$e^{at} f(t)$	$F(s-a)$
(20)	$f'(t)$	$sF(s)$
(21)	$tf(t)$	$-\frac{d}{ds} F(s)$
(22)	$f * g(t)$	$F(s)G(s)$
(23)	$\theta(t) f'(t)$	$s\,\mathcal{L}_{II}(\theta f)(s) - f(0)$
(24)	$\delta(t)$	1
(25)	$\theta(t)$	$\frac{1}{s},\ \sigma > 0$
(26)	$\theta(t) - 1$	$\frac{1}{s},\ \sigma < 0$
(27)	$t^k e^{at} \theta(t)$	$\frac{k!}{(s-a)^{k+1}},\ \sigma > \operatorname{Re} a$
(28)	$\sin(bt)\theta(t)$	$\frac{b}{s^2+b^2},\ \sigma > 0$
(29)	$\cos(bt)\theta(t)$	$\frac{s}{s^2+b^2},\ \sigma > 0$
(30)	e^{-t^2}	$\sqrt{\pi} e^{s^2/4}$
(31)	$t^{\alpha-1}\theta(t)$	$\frac{\Gamma(\alpha)}{s^\alpha},\ \operatorname{Re}\alpha > 0,\ \operatorname{Re} s > 0$
(32)	$\frac{\lvert a\rvert}{\sqrt{4\pi}} \frac{e^{-a^2/4t}}{t^{3/2}} \theta(t)$	$e^{-\lvert a\rvert\sqrt{s}}$
(33)	$\frac{1}{\sqrt{\pi t}} e^{-a^2/4t} \theta(t)$	$\frac{e^{-\lvert a\rvert\sqrt{s}}}{\sqrt{s}}$