Linear and Combinatorial Optimization

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1 Duality
The weak duality theorem

Recall the weak duality theorem and some of its consequences from last time:

**Theorem (The weak duality theorem)**

If $x$ is a feasible solution of $(P)$ and $y$ is a feasible solution of $(D)$, then $c^T x \leq b^T y$.

**Corollary**

If the primal problem is unbounded, then the dual problem is infeasible.

Figure 1: Weak duality theorem
The weak duality theorem

Theorem

If \( x \) and \( y \) are feasible solutions of the primal and dual problem, respectively, and if \( c^T x = b^T y \), then both \( x \) and \( y \) are optimal solutions of their respective problems.

- Note: It can happen that both the primal and dual problems are infeasible.
- If the dual problem is unbounded (to \( -\infty \)), then the primal problem is infeasible (since the dual of the dual is the primal!).
The strong duality theorem

**Theorem (The strong duality theorem)**

If $\hat{x}$ is optimal and feasible for (P), then there exists a $\hat{y}$ which is optimal and feasible for (D), and $c^T\hat{x} = b^T\hat{y}$.

**Proof.**

Introduce slack variables to put (P) into canonical form, and solve the problem with the simplex algorithm. There exists an optimal solution $\hat{x} = \begin{bmatrix} x \\ x' \end{bmatrix}$, where $x'$ is the vector of slack variables. Let $\hat{c} = \begin{bmatrix} c \\ 0 \end{bmatrix}$.

Figure 2: Strong duality theorem
Proof (Cont.)

We decompose \(\hat{x}\) into its basic and nonbasic variables, \(\hat{x}_B\) and \(\hat{x}_N = 0\), and change the order of the variables so that \(\hat{x} = \begin{bmatrix} \hat{x}_N \\ \hat{x}_B \end{bmatrix}\).

At the same time, we need to change the order of the columns in \([A \ I]\) and in \(\hat{c}\). Then \(\hat{x}_B = B^{-1}b\), where \(B\) is the (permuted!) submatrix of \([A \ I]\) corresponding to the basic variables of the optimal solution. Also, we decompose \(\hat{c}\) into \(\hat{c}_B\) and \(\hat{c}_N\), and let \(y = (B^{-1})^T\hat{c}_B\).
The strong duality theorem (Cont.)

Proof (Cont.)

Then \( y^T = \hat{c}_B^T B^{-1} \), and so

\[
z = \hat{c}^T \hat{x} = \hat{c}_N^T \cdot 0 + \hat{c}_B^T \hat{x}_B = \hat{c}_B^T B^{-1} b = y^T b = b^T y.
\]

By the weak duality theorem, we are done if we can show that \( y \) is a feasible solution of (D). Last time, we used the simplex method for a problem in canonical form:

\[
\begin{cases}
\text{maximize} & z = c^T x, \\
\text{subject to} & A x = b, \\
& x \geq 0.
\end{cases}
\]
The strong duality theorem (Cont.)

Proof (Cont.)

We decomposed $A$ into $[A_N \ A_B]$, $x$ into $[x_N \ x_B]$ and $c$ into $[c_N \ c_B]$ and solved for $x_B$ in terms of $x_N$. We got the tableau

$$
\begin{array}{c|c}
\hat{x}_B & A_B^{-1}A_Nx_N + x_B = A_B^{-1}b \\
\hat{x}_B & B^{-1}N\hat{x}_N + \hat{x}_B = B^{-1}b \\
\end{array}
$$

$$
\begin{array}{c|c}
z & (c_B^TA_B^{-1}A_N - c_N^T)x_N + z = c_B^TA_B^{-1}b \\
z & (\hat{c}_B^TB^{-1}N - \hat{c}_N^T)\hat{x}_N + z = \hat{c}_B^TB^{-1}b \\
\end{array}
$$

If we do the same for our problem, and denote the (permutated) submatrix of the original matrix $[A \ I]$ corresponding to the nonbasic variables of the optimal solution $\hat{x}$ by $N$, the tableau becomes

$$
\begin{array}{c|c}
\hat{x}_B & B^{-1}N\hat{x}_N + \hat{x}_B = B^{-1}b \\
\hat{x}_B & B^{-1}N\hat{x}_N + \hat{x}_B = B^{-1}b \\
\end{array}
$$

$$
\begin{array}{c|c}
z & (c_B^TB^{-1}N - c_N^T)\hat{x}_N + z = \hat{c}_B^TB^{-1}b \\
z & (\hat{c}_B^TB^{-1}N - \hat{c}_N^T)\hat{x}_N + z = \hat{c}_B^TB^{-1}b \\
\end{array}
$$
The solution \( \hat{x} \) is optimal if and only if

\[
\hat{c}_B^T B^{-1} N - \hat{c}_N^T \geq 0,
\]

according to the optimality criterion in the simplex algorithm. But note that we also have

\[
\hat{c}_B^T B^{-1} B - \hat{c}_B^T = \hat{c}_B^T - \hat{c}_B^T = 0 \geq 0 \text{ (trivially!)}
\]

Together, this gives

\[
\hat{c}_B^T B^{-1} \begin{bmatrix} N & B \end{bmatrix} - \begin{bmatrix} \hat{c}_N^T & \hat{c}_B^T \end{bmatrix} \geq 0.
\]

But \( \begin{bmatrix} N & B \end{bmatrix} \) is just the matrix \( \begin{bmatrix} A & I \end{bmatrix} \) with permuted columns, and \( \begin{bmatrix} \hat{c}_N^T & \hat{c}_B^T \end{bmatrix} \) is the same permutation of the row vector \( \begin{bmatrix} c^T & 0 \end{bmatrix} \).
The strong duality theorem (Cont.)

Proof.

Hence

$$\hat{c}_B^T B^{-1} [A \quad I] - [c^T \quad 0] \geq 0$$

which is equivalent to

$$\begin{cases} \hat{c}_B^T B^{-1} A \geq c^T, \\ \hat{c}_B^T B^{-1} \geq 0. \end{cases}$$

Recall that $y = (B^{-1})^T \hat{c}_B$. So (*) is equivalent to

$$\begin{cases} y^T A \geq c^T, \\ y^T \geq 0. \end{cases} \iff \begin{cases} A^T y \geq c, \\ y \geq 0, \end{cases}$$

which shows that $y$ is feasible for (D).
Complementary slackness

Definition

Let $x$ and $y$ be feasible solutions for (P) and (D), respectively. Then $x$ and $y$ are said to satisfy the complementary slackness condition (CS) if

$$y^T(Ax - b) = 0 \quad \text{and} \quad x^T(A^Ty - c) = 0.$$ 

What does this mean?

Recall that $y \geq 0$ and the slack variables $x' = b - Ax \geq 0$. We have

$$y^T(Ax - b) = 0 \iff -y^Tx' = 0 \iff y^Tx' = 0 \iff y_jx'_j = 0 \text{ for all } j.$$
Complementary slackness

Since all the terms $y_j x'_j \geq 0$, this is equivalent to $y_j x'_j = 0$ for all $j = 1, \ldots, m$, i.e. if and only if $y_j = 0$ or $x'_j = 0$ for all $j = 1, \ldots, m$. This proves that

$$y^T (Ax - b) = 0 \iff y_j = 0 \text{ or } (Ax)_j = b_j \text{ for every } j = 1, \ldots, m.$$

If for some $j \in \{1, \ldots, m\}$, $(Ax)_j = b_j$, we say that the $j$th constraint is active.

In the same way as above, we can show that

$$x^T (A^Ty - c) = 0 \iff x_i = 0 \text{ or } (A^Ty)_i = c_i \text{ for every } i = 1, \ldots, n.$$
Complementary slackness

Lemma

If \( x, y \) satisfy (CS), then \( c^T x = b^T y \)
(and so \( x \) and \( y \) are optimal for (P) and (D), respectively, by the weak duality theorem).

Proof.

\[(CS) \quad \implies \quad y^T Ax = y^T b = b^T y,
\]

but also

\[ y^T Ax = x^T A^T y = x^T c = c^T x, \]

and so

\[ c^T x = b^T y. \]
Complementary slackness

**Theorem**

If \( x \) is optimal for \((P)\) and \( y \) is optimal for \((D)\), then \((CS)\) holds.

**Proof.**

By the strong duality theorem, we have \( c^T x = b^T y \). Introduce slack variables for \((P)\) so that

\[
x' = b - Ax \iff x'^T = b^T - x^T A^T
\]

which implies that

\[
x'^T y = b^T y - x^T A^T y \leq c^T x - x^T c = 0.
\]

Hence \((b - Ax)^T y \leq 0\). But \( b - Ax \geq 0 \) and \( y \geq 0 \), and so equality holds. In the same way, it can be proved that

\[
x^T (A^T y - c) = 0.
\]
The diet problem

Note that we can solve whichever problem is easier to solve of (P) and (D), and then we automatically get a solution also for the other problem.

Example (Diet problem, p. 46–47 in Kolman–Beck)

2 foods, $F_1$ and $F_2$ contain nutrients $N_1$, $N_2$, and $N_3$. The nutrient content and price per unit of food is given in the table below together with the minimal amounts required for each unit.

<table>
<thead>
<tr>
<th></th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>$F_2$</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>min amount</td>
<td>18</td>
<td>12</td>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>
The diet problem

Example (Diet problem, Cont.)

How do we compose a meal satisfying the nutrient requirements for the smallest possible cost?

This can be formulated as an LP problem as follows: Let \( x_1, x_2 \) be the amounts of the foods \( F_1 \) and \( F_2 \) that goes into the meal. The LP problem is then the minimization problem

\[
\text{minimize} \quad z = 20x_1 + 25x_2,
\]

subject to

\[
\begin{align*}
2x_1 + 3x_2 &\geq 18, \\
x_1 + 3x_2 &\geq 12, \\
4x_1 + 3x_2 &\geq 24, \\
x_1, x_2 &\geq 0.
\end{align*}
\]
The diet problem

Example (Diet problem, Cont.)

Now, let’s say that there is a manufacturer of artificial foods $P_1$, $P_2$, $P_3$, where one unit of $P_j$ contains one unit of $N_j$. How should the manufacturer set the prices $y_1$, $y_2$, $y_3$ of the foods $P_1$, $P_2$, $P_3$? The cost of the substitute for $F_j$ cannot be higher than the cost of $F_j$ (otherwise nobody would buy it). This gives the constraints

\[
\begin{align*}
2y_1 + y_2 + 4y_3 & \leq 20, \\
3y_1 + 3y_2 + 3y_3 & \leq 25, \\
y_1, y_2, y_3 & \geq 0.
\end{align*}
\]

The profit should be maximized, so the problem is to

\[
\text{maximize } v = 18y_1 + 12y_2 + 24y_3
\]

subject to the above constraints.
Note that this is precisely the dual problem of the diet problem. We can choose to solve either of them. We notice that in the dual problem, phase 1 of the two-phase method is not required since the right hand side of the constraint vector has only positive entries.

Solving this with the simplex method, we get $y_1 = \frac{20}{3}$, $y_2 = 0$, $y_3 = \frac{5}{3}$ and slack variables $y_4 = 0$, $y_5 = 0$.

The complementary slackness condition implies that constraint number 1 and 3 are active in $(P)$. Hence

\[
\begin{aligned}
2x_1 + 3x_2 &= 18 \\
4x_1 + 3x_2 &= 24,
\end{aligned}
\]
Example (Diet problem, Cont.)

- The above system has the solution $x_1 = 3$ and $x_2 = 4$.
- The optimal value for $(P)$ is $20 \cdot 3 + 25 \cdot 4 = 160$, and for $(D)$ it is $18 \cdot \frac{20}{3} + 24 \cdot \frac{5}{3} = 6 \cdot 20 + 8 \cdot 5 = 160$, which are the same as expected.