Calculus of Variations
lecture 5
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Mark Kot: Ch. 4.3 p 77-78, Ch. 1.3 p.7-74
ch. 9.1 p. 191-197.

(Hesterton-Gibbons: Lecture 7, p. 57-56)

Suggested exercises on separate sheet
Application: Finding geodesics

We want to determine the shortest path connecting two points \( P_A \) and \( P_B \) on a surface \( \Sigma \) in \( \mathbb{R}^3 \):

Let \( \bar{F}(u,v) = (x(u,v), y(u,v), z(u,v)) \) be a parametrization of \( \Sigma \).
If \( \alpha(t) = (u(t), v(t)) \) defines a curve in the parameter domain, then

\[
\gamma(t) = \bar{F}(u(t), v(t)) = \bar{F} \circ \alpha(t)
\]
gives a curve in \( \Sigma \). The speed of the curve is

\[
\dot{\gamma}(t) = \bar{F}_u \cdot \dot{u}(t) + \bar{F}_v \cdot \dot{v}(t)
\]

so the length of the curve element covered in a time interval of length \( dt \) is

\[
|\dot{\gamma}(t)| dt = |\bar{F}_u \cdot \dot{u}(t) + \bar{F}_v \cdot \dot{v}(t)| dt
\]

\[
= \sqrt{|\bar{F}_u|^2 \dot{u}^2 + 2 \bar{F}_u \cdot \bar{F}_v \dot{u} \dot{v} + |\bar{F}_v|^2 \dot{v}^2} \, dt
\]
If $s$ denotes curvilinear length, then the previous expression is usually written as
\[
ds^2 = |\mathbf{F}_u|^2 du^2 + 2 \mathbf{F}_u \cdot \mathbf{F}_v du dv + |\mathbf{F}_v|^2 dv^2.
\]
In differential geometry, this is referred to as the first fundamental form.

To find shortest paths corresponds to the minimization of
\[
L[(u,v)] = \int_0^1 \sqrt{|\mathbf{F}_u|^2 u^2 + 2 \mathbf{F}_u \cdot \mathbf{F}_v u v + |\mathbf{F}_v|^2 v^2} \, dt
\]
subject to $(u(0), v(0)) = A$ and $(u(1), v(1)) = B$. (See figure).

Example (Shortest paths on the cylinder)

$\mathbf{F}(\varphi, \xi) = (r \cos \varphi, r \sin \varphi, \xi)$,

$0 \leq \varphi < 2\pi$, \text{ and } $0 \leq \xi \leq R$, parameterizes a right cylinder of radius $r$.

$\mathbf{F}_\varphi = (-r \sin \varphi, r \cos \varphi, 0)$

$\mathbf{F}_\xi = (0, 0, 1)$

\[
ds^2 = |\mathbf{F}_\varphi|^2 d\varphi^2 + 2 \mathbf{F}_\varphi \cdot \mathbf{F}_\xi d\varphi d\xi + |\mathbf{F}_\xi|^2 d\xi^2
\]
(Cont'd.)

\[ ds^2 = r^2 d\phi^2 + d\xi^2 \]

We assume \( A : (0,0) \) and \( B : (\xi_0,0) \) are the parameters of \( P_A \) and \( P_B \) on the cylinder.

We are asked to find a curve \( \vec{\alpha}(t) = (\phi(t), \xi(t)) \), \( 0 \leq t \leq 1 \), in the parameter domain which minimizes

\[ J[\vec{\alpha}] = \int_0^1 \sqrt{r^2 \dot{\phi}(t)^2 + \dot{\xi}(t)^2} \, dt \]

subject to \( \vec{\alpha}(0) = (0,0) \) and \( \vec{\alpha}(1) = (\phi_0, \xi_0) \).

The Euler eqs are

\[
\frac{d}{dt} \left\{ \frac{r^2 \dot{\phi}}{\sqrt{r^2 \phi^2 + \xi^2}} \right\} = 0,
\frac{d}{dt} \left\{ \frac{\dot{\xi}}{\sqrt{r^2 \phi^2 + \xi^2}} \right\} = 0
\]

\[
\frac{r^2 \phi}{\sqrt{r^2 \phi^2 + \xi^2}} = \text{const.} \quad (\text{Assumed}) \quad \frac{\dot{\xi}}{\sqrt{r^2 \phi^2 + \xi^2}} = \text{const}
\]

In particular \( \phi \) always has the same sign. If we divide the second eq. by the first, then we find

\[
\frac{d\xi}{d\phi} = \frac{\dot{\xi}}{\dot{\phi}} = \text{const} = A, \quad A \in \mathbb{R}
\]

\[ \xi(\phi) = A\phi + B \]

An admissible extremal is

\[ \xi(\phi) = \xi_0 \frac{\phi}{\phi_0} \]
We consider the minimization of
\[ J[\gamma] = \int_a^b F(x, \gamma, \gamma') \, dx, \]
where \( \gamma \in C_1 \), but without requirements on the values \( \gamma(a) \) and \( \gamma(b) \). Such a problem is called a variable end point problem.

Under the assumption that \( \phi \in C_2 \) (Obs!) minimizes \( J \), we show that in addition to Euler's equation, \( \phi \) will also satisfy certain additional conditions at \( x = a \) and \( x = b \). These are called natural boundary conditions.

So, suppose that \( J[\phi] \leq J[\gamma] \) for all \( \gamma \in C_1 \), and \( \phi \in C_2 \). Define
\[ \gamma_\varepsilon(x) = \phi(x) + \varepsilon \eta(x), \]
where \( \eta \in C_1 \). Observe that \( \eta(a) \) and \( \eta(b) \) may now have any value, not only zero. If we define \( J(\varepsilon) = J[\gamma_\varepsilon] \), then (as usual)
\[ 0 = J'(0) = \int_a^b \left( F_\phi \eta + F_{\phi'} \eta' \right) \, dx. \]

The assumption \( \phi \in C_2 \) implies \( F_{\phi'} \in C_1 \) so we may integrate by parts:
\[ 0 = \int_a^b \left( F_{\phi'} - \frac{d}{dx} F_\phi \right) \eta \, dx + \left[ F_\phi \eta \right]_a^b. \]

We have great freedom in choosing the variation \( \eta \in C_1 \). If we first consider...
variations with \( h(a) = h(b) = 0 \), then

\[
0 = \int_a^b \left\{ F_\phi - \frac{d}{dx} F_\phi' \right\} \phi \, dx,
\]

so we find Euler's equation:

\[
(E_\phi) \quad F_\phi - \frac{d}{dx} F_\phi' = 0, \quad a < x < b,
\]
as a consequence of the fundamental lemma. In view of \((E_\phi)\) the condition (2) becomes

\[
0 = [F_\phi, h]_a^b = F_\phi' \big|_{x=a} h(a) - \frac{d}{dx} \left[ F_\phi' \big|_{x=b} h(b) \right].
\]

Since \( h(a) \) and \( h(b) \) can be chosen independently of each other, we find the following natural boundary cond.

\[
(NB) \quad \begin{cases} 
F_{\phi}(a, \phi(a), \phi'(a)) = 0, \\
F_{\phi}(b, \phi(b), \phi'(b)) = 0.
\end{cases}
\]

**Example** (The Brachistochrone, again)

We want to determine the shape of a slide along which from \((0,0)\) to the vertical line \(x = b\) along which a bead, under the influence of gravity, will fall in the shortest possible time:

\[
T = \int_0^b \frac{\sqrt{1 + y'^2}}{\sqrt{y}} \, dx, \quad y(0) = 0.
\]
We already know that the extremals (i.e. solutions of Euler's eq.) are the cycloids through the origin:

\[
(\text{Ext.}) \quad \begin{cases} 
    x = \frac{A}{2} (\varphi - \sin \varphi) \\
    y = \frac{A}{2} (1 - \cos \varphi)
\end{cases} \quad 0 \leq \varphi \leq 2\pi
\]

Also \( \frac{y'}{\sqrt{y'^2 + 1}} \) so the natural boundary condition at \( x = b \) is

\[
F_y' (b, y(b), y'(b)) = 0 \iff y'(b) = 0.
\]

To use this condition, we have to deal with the fact that the extremals (\text{Ext.}) are given in parametric form.

We have to determine \( A > 0 \) such that for some \( \varphi_0 \), \( 0 < \varphi_0 < 2\pi \), we have

\[
\begin{cases} 
    \varphi (\varphi_0) = b \quad \text{and} \\
    \dot{\varphi} (\varphi_0) = 0 \quad (\dot{\varphi} = \frac{d}{d \varphi})
\end{cases}
\]

The latter because \( 0 = y'(b) = \frac{\dot{y}(\varphi_0)}{x(\varphi_0)} \), by the chain rule. Since

\[
\dot{y} = \frac{A}{2} \sin \varphi = 0, \quad 0 < \varphi < 2\pi \iff \varphi_0 = \pi
\]

we get \( b = x(\pi) = \frac{A}{2} \pi \), thus

\[
\begin{cases} 
    x = \frac{b}{\pi} (\varphi - \sin \varphi) \\
    y = \frac{b}{\pi} (1 - \cos \varphi)
\end{cases} \quad 0 \leq \varphi \leq \pi.
\]
A simple example

(P) \[ \begin{align*} \text{Minimize } & J = \int_0^7 \frac{1}{2} y'(x)^2 + y(x)y'(x) \, dx \\ \text{Subject to } & y(0) = 2 \end{align*} \]

In (P), the right end-point at \(x=7\) is free. This means that the necessary condition for optimality becomes

\[ \begin{cases} F_y - \frac{d}{dx} F_y' = 0, \quad F = \frac{1}{2} y'^2 + y y' \quad (\text{Euler}) \\ y(0) = 2 \\ F_y, (y(t), y'(t)) = 0 \quad (\text{end point}) \end{cases} \]

That is, since \( F_y = y' \), \( F_y' = y' + y \), we get

\[ \begin{cases} y' - \frac{d}{dx} (y' + y) = y'' = 0 \\ y(0) = 2 \\ y'(7) + y(7) = 0 \end{cases} \]

\[ \begin{aligned} y(x) &= Ax + B \\ y(0) = 1 &\Rightarrow B = 2 \\ 0 &= y'(7) + y(7) = A + A + 2 \iff A = -1 \end{aligned} \]

so the admissible extremal is

\[ \phi(x) = -x + 2, \quad 0 \leq x \leq 7. \]

Prove that \( \phi \) is a minimizer, using a direct verification. \( J[\phi] = -1 \).