Calculus of variations
2012-02-09
Lecture 6
Nicks Curr. Overgaard

The Weierstrass-Erdmann conditions

\begin{align*}
\text{Holtz Köt: Ch. 10, pp. 215-221} \\
\text{Hesterton-Gibbons: Lecture 6, pp. 41-50}
\end{align*}
D-1 - Theory

Some variational problems have natural solutions if we admit piecewise differentiable functions.

Example The functional \( J[y] = \int (x^2 - y'^2) \, dx \)
is minimized in \( D_1 \) subject to \( y(-1) = y(1) = 1 \)
by \( \varphi(x) = |x| \), which is not in \( C_1 \). That is, \( \varphi \) is a broken extremal.

In the second part of the lecture we want to show that a broken extremals

1) Satisfies Euler's equation between corners

2) Satisfies the two Weierstrass-Erdmann conditions at corners.

Recall of Definitions

The class \( D \) of piecewise continuous functions. A function \( u : [a, b] \rightarrow \mathbb{R} \)
belongs to \( D \) if there exists numbers

\[
a = x_0 < x_1 < x_2 < \ldots < x_n = b
\]
such that \( u \) is continuous on \([x_{i-1}, x_i)\)
for \( i = 1, \ldots, N \). Moreover, \( \lim_{x \to x_i^-} u(x) \) exists.

(See fig.:)

The class \( D_1 \) of piecewise differentiable functions consist of continuous functions \( u \)
which are differentiable except at a finite number of points, and such that \( u' \in D \).
du Bois-Reymond's lemma and its corollary hold in variants where continuous is replaced by piecewise continuity:

**Lemma.** Suppose $M \in D_1$ satisfies

$$\int_a^b M(x) \gamma'(x) \, dx = 0$$

for all $\gamma \in D_1$ such that $\gamma(b) = \gamma(a) = 0$. Then $M(x) = C'$, $a \leq x \leq b$, for some constant $C$.

**Corollary.** Suppose $M, N \in D_1$ satisfy

$$\int_a^b M \gamma' + N \gamma \, dx = 0$$

for all $\gamma \in D_1$ with $\gamma(b) = \gamma(a) = 0$. Then

$$M(x) = \int_a^x N'(x') \, dx' + C,$$

for some constant $C \in \mathbb{R}$. In particular, $M \in D_1$ with $M'(x) = N(x)$. (What happens at any of the finite number of points where the derivative of $M$ does not exist?)

**Remark.** The proofs are essentially the same as before. Two points should be noticed: If $\gamma \in D_1$, then

$$\int_a^b \gamma'(x) \, dx = \gamma(b) - \gamma(a)$$

still holds. (Prove this!) Also, if $\gamma, \phi \in D_1$, then

$$\int_a^b \phi \gamma' \, dx = \left[ \phi \gamma \right]_a^b - \int_a^b \phi' \gamma \, dx.$$
Application to the standard problem

If \( \phi \in D_1 \) solves the problem of minimizing

\[ J[\gamma] = \int_a^b F(x, y, y') \, dx \]

subject to \( y \in D_1 \) with \( y(a) = \alpha \) and \( y(b) = \beta \), then \( \phi' \in D \) and the functions

\[ F_\phi(x) = F_y(x, \phi(x), \phi'(x)) \quad \text{and} \]
\[ F_{\phi'}(x) = F_{y'}(x, \phi(x), \phi'(x)) \]

are likewise in \( D \). (Here we assume that the Lagrange function \( F \) is sufficiently regular.)

We still have

\[ \int_a^b F_\phi \eta + F_{\phi'} \eta' \, dx = 0, \]

now for all variations \( \eta \in D_1 \) with \( \eta(b) = \eta(a) = 0 \).

The corollary above yields du Bois-Reymond's

\[ (\text{dBR}) \quad F_{\phi'}(x) = \int_a^x F_\phi(\xi) \, d\xi + C, \quad a \leq x \leq b. \]

Consequence 1 (Euler's eq.) Between corners the integrand \( F_\phi(\xi) \) in (dBR) is continuous, and so \( F_{\phi'}(x) \) is differentiable, and

\[ \frac{d}{dx} F_{\phi'} = F_{\phi}. \]
Consequence 2. Suppose \( y = \phi(x) \) has a corner at \( x = c \). This means that 
\[
\omega_1 = \lim_{x \to c^-} \phi'(x) \quad \text{and} \quad \omega_2 = \lim_{x \to c^+} \phi'(x)
\]
satisfy
\[
\omega_1 \neq \omega_2
\]

Now, the right hand side of (DBK) is continuous, corner or not. It follows that \( F_y'(x) \) is also continuous at \( x = c \), so

\[
(FE-1) \quad F_y'(c, \phi(c), \omega_1) = F_y'(c, \phi(c), \omega_2)
\]

This is the first Weierstrass-Erdmann condition.

Example Once again consider the functional

\[
J[y] = \int_{-1}^{1} (1-y'^2)^2 \, dx
\]

minimized subject to \( y(-1) = y(1) = 1 \).

The function \( \phi(x) = |x| \) solves this problem and has a corner at \( x = 0 \) where \( \omega_1 = \phi'(0^-) = -1 \) and \( \omega_2 = \phi'(0^+) = 1 \).

Now, \( F_y' = -4y'(1-y'^2) \), and

\[
F_y'(0,0,-1) = 0 = F_y'(0,0,1),
\]

so the first Weierstrass-Erdmann condition holds.
Example (cont'd.) Let us examine the corner conditions a little more closely, to see which pairs \((\omega_1, \omega_2)\) are allowed.

\[
F_y'(c, \phi(c), \omega_1) = F_y'(c, \phi(c), \omega_2)
\]

\[
\Leftrightarrow \quad \omega_1(1 - \omega_1^2) = \omega_2(1 - \omega_2^2)
\]

\[
\Leftrightarrow \quad (\omega_1 - \omega_2) - (\omega_1^3 - \omega_2^3) = 0
\]

\[
(\omega_1 - \omega_2)(1 - \omega_1^2 - \omega_1 \omega_2 - \omega_2^2) = 0
\]

The latter equation holds if \(\omega_1 = \omega_2 = 0\) (no corner!) or if

\[
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 = 1
\]

which is an ellipse:

Clearly \((\omega_1, \omega_2) = (-1, 1)\)

and \((\omega_1, \omega_2) = (1, -1)\) lie on the ellipse, but there are many more solutions! ... This indicates that a condition is missing!!
The Second Weierstrass - Erdmann Condition

Define \( H(x, y, y') = y' F_y - F \). The second Weierstrass - Erdmann condition states that along an extremal \( y = \phi(x) \in D \), the function \( H(x, \phi(x), \phi'(x)) \) is continuous.

At a corner \( x = c \) of the broken extremal \( y = \phi(x) \) this means that

\[(WE-2) \quad H(c, \phi(c), \omega_1) = H(c, \phi(c), \omega_2).\]

We will prove this result in the next lecture.

Example Returning to the previous example with \( F = (-1-y'^2)^2 \) we found \( F_y = -4y'(-1-y'^2) \) and therefore

\[
H(x, y, y') = y' F_y - F = -4y'^2(1-y'^2) - (1-y'^2)^2 = (1-y'^2)(-3y'^2 - 1).
\]

Thus (WE-2) states that

\[
(1-\omega_1^2)(1+3\omega_2^2) = (1-\omega_2^2)(1+3\omega_2^2)
\]

\[
2(\omega_1^2 - \omega_2^2) - 3(\omega_1^4 - \omega_2^4) = 0
\]

\[
(\omega_1 - \omega_2)[2(\omega_1 + \omega_2) - 3(\omega_1^3 + \omega_2^3 + \omega_1 \omega_2^2 + \omega_2 \omega_1^2)] = 0
\]
Example (contd.) We want to solve the system of equations given by (WE-1) and (WE-2) simultaneously:

\[
\begin{cases}
(\omega_1 - \omega_2)(\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1) = 0 \\
(\omega_1 - \omega_2)[2(\omega_1 + \omega_2) - 3(\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 + \omega_3^3)] = 0
\end{cases}
\]

We don't want \( \omega_1 = \omega_2 \) (no corner) so we need only be concerned with zeroes of the second factors:

\[
\begin{cases}
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1 = 0 \\
2(\omega_1 + \omega_2) - 3(\omega_1^3 + \omega_1^2 \omega_2 + \omega_1 \omega_2^2 + \omega_2^3) = 0 \\
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1 = 0 \\
2(\omega_1 + \omega_2) - 3(\omega_1 + \omega_2)(\omega_1^2 + \omega_1 \omega_2 + \omega_2^2) + 3(\omega_1 + \omega_2)\omega_1 \omega_2 = 0
\end{cases}
\]

\[
\begin{cases}
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1 = 0 \quad (\text{Using that } \omega_1^2 + \omega_1 \omega_2 + \omega_2^2 = 1) \\
(\omega_1 + \omega_2)(3\omega_1 \omega_2 - 1) = 0
\end{cases}
\]

\[
\begin{cases}
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1 = 0 \quad (\text{Using } \omega_1 \omega_2 = 1 = -\omega_1^2 - \omega_2^2) \\
(\omega_1 + \omega_2)(2\omega_1 \omega_2 - \omega_1^2 - \omega_2^2) = 0
\end{cases}
\]

\[
\begin{cases}
\omega_1^2 + \omega_1 \omega_2 + \omega_2^2 - 1 = 0 \\
(\omega_1 + \omega_2)(\omega_1 - \omega_2)^2 = 0
\end{cases}
\]

So, disregarding solutions with \( \omega_1 = \omega_2 \), we get \( (\omega_1, \omega_2) = (1, -1) \) or \( (-1, 1) \).
A geometric interpretation of the two Weierstrass-Erdmann Corner Conditions

We know that if a functional

$$J[y] = \int_a^b F(x, y(x), y'(x)) \, dx$$

admits a minimizer $\phi \in D_1$ with a corner at some point $c \in (a, b)$, then we must necessarily have (WE-1),

$$F_y, (c, \phi(c), \phi'(c-0)) = F_y, (c, \phi(c), \phi'(c+0))$$

and (WE-2),

$$F(c, \phi(c), \phi'(c-0)) - \phi'(c-0)F_y, (c, \phi(c), \phi'(c-0)) = F(c, \phi(c), \phi'(c+0)) - \phi'(c+0)F_y, (c, \phi(c), \phi'(c+0))$$

where $\phi'(c-0) = \lim_{x \to c^-} \phi'(x)$ and $\phi'(c+0) = \lim_{x \to c^+} \phi'(x)$.

If we introduce the so-called characteristic of $F$ (sometimes called the indicatrix of $F$) by

$$F(\omega) := F(c, \phi(c), \omega), \quad \omega \in \mathbb{R}$$

and set $\omega_1 = \phi'(c-0)$ and $\omega_2 = \phi'(c+0)$, then the corner conditions become

$$\begin{cases}
F'(\omega_1) = F'(\omega_2) \\
F(\omega_1) - \omega_1 F'(\omega_1) = F(\omega_2) - \omega_2 F'(\omega_2)
\end{cases}$$
If we introduce $H(\omega) = \omega F'(\omega) - F(\omega)$, then the second corner condition may be written compactly as

$$H(\omega_1) = H(\omega_2).$$

Now, if $\phi$ has a corner at $x = c$ then $\omega_1 \neq \omega_2$, and we may give a neat geometrical interpretation of the two corner conditions.

The tangent lines at $\omega_1$ and $\omega_2$ of the characteristic $F(\omega)$ must coincide.

\[ y = F(\omega) \]

**Pf.** Once noticed the proof is quite simple. The tangent line through $(\omega_1, F(\omega_1))$ is given by

$$y = F(\omega_1) + F'(\omega_1)(\omega - \omega_1)$$

$$= F'(\omega_1)\omega - H(\omega_1)$$

and that through $(\omega_2, F(\omega_2))$ by

$$y = F'(\omega_2)\omega - H(\omega_2).$$

By the corner conditions they are the same.