Calculus of variations

Lecture 9

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Topics

1. Some consequences of the corner conditions.

2. Proof of the second Weierstrass-Erdmann corner condition.

3. Legendre's necessary condition.
We also stated, but did not prove, the second Weierstrass-Erdmann condition:

\[(\text{WE-2}) \quad H(c, \Phi(c), \omega_1) = H(c, \Phi(c), \omega_2)\]

where \(H(x, y, y') = y' F_y \) if \(F\) is the Hamiltonian.

**Some Consequences** Before turning to the proof of (\text{WE-2}) we observe some useful consequences of (\text{WE-1}) and (\text{ABB}).

1. If the Lagrange function \(F(x, y, y')\) is such that \(y' \mapsto F(x, y, y')\) is strictly monotone, then

\[F_y(x, y(c), \omega_1) = F_y(x, y(c), \omega_2)\]

is impossible unless \(\omega_1 = \omega_2\) (no corner).

**Example.** The functional \(\int_{-1}^{1} y'(x)^4 \, dx\) has \(F = y'^4\) so \(F_y = 4y'^3\), which is strictly monotone. Therefore this functional does not admit any broken extremals. (The only extremal being straight line segments \(y(x) = Ax + B\), \(-1 \leq x \leq 1\).)

2. Suppose \(F_y y'(x, y, y') > 0\) for all \((x, y, y')\), then, in particular, \(y' \mapsto F_y(x, y, y')\) is strictly mono-
tone. By \( T \), the functional
\[
J[y] = \int_{a}^{b} F(x, y, y') \, dx \quad (\star)
\]
does not allow broken extremals. However, \( Fy'y' > 0 \) is a stronger condition than \( y' \to Fy \) being strictly monotone, so we can get more: If \( \phi \) minimizes \( (\star) \), then \( \phi \in C_1 \), by the argument above, in fact, \( \phi \in C_2 \) by the following argument:

Define \( G : [a, b] \times \mathbb{R} \to \mathbb{R} \) by
\[
G(x, p) = Fy'(x, \phi(x), p) - C
- \int_{a}^{x} Fy(\xi, \phi(\xi), \phi'(\xi)) \, d\xi.
\]
Then \( G \in C_1 \) because \( \phi \in C_1 \) and \( F \) is smooth. By \((\text{ABR})\) we know that the eq.
\[
(\star) \quad G(x, p) = 0
\]
is solved by
\[
p = \phi'(x), \quad a \leq x \leq b.
\]
Since \( G_p(x, p) = Fy'y'(x, \phi(x), p) > 0 \), it follows from the implicit function theorem that the solution \( p = p(x) \) of \( (\star) \) is \( C_1 \). But \( p = \phi' \in C_1 \) implies \( \phi \in C_2 \).
Proof of the second corner condition

Suppose \( \varphi(x) \) minimizes the functional

\[
I[y] = \int_a^b F(x, y(x), y'(x)) \, dx
\]

subject to the boundary conditions \( y(a) = \alpha \) and \( y(b) = \beta \). We assume only that \( \varphi \in D_1 \), so it may be a broken extremal.

The goal is to prove that the function

\[
\varphi \mapsto F(x, \varphi(x), \varphi'(x)) - \varphi'(x) F_{y'}(x, \varphi(x), \varphi'(x))
\]

is continuous on \( a \leq x \leq b \).

In particular, if \( \varphi \) has a corner at \( x = c \) with \( \varphi'(c+) = \omega_+ \) and \( \varphi'(c-) = \omega_- \) then the second WE condition holds

\[
F(c, \varphi(c), \omega_-) - \omega_- F_{y'}(c, \varphi(c), \omega_-) = F(c, \varphi(c), \omega_+) - \omega_+ F_{y'}(c, \varphi(c), \omega_+).
\]

(Clearly it is easier to say that \( F - \varphi' F_{y'} \) is continuous at \( c \).)

The proof that is presented here is from L.A. Pars' book "An introduction to the calculus of variations", Dover 2010.

We consider the curve \( y = \varphi(x) \) as a parametrized curve,

\[
\gamma_0: \quad x = t, \quad y = \varphi(t), \quad a \leq t \leq b.
\]
We take a function \( \lambda \in D_1 \) with \( \lambda(a) = \lambda(b) = 0 \) and define a one-parameter family of curves:

\[ y_\alpha : \quad x = t + \alpha \lambda(t), \quad y = \varphi(t), \quad a \leq t \leq b, \]

where \( \alpha \) is a real parameter.

Thus \( y_\alpha \) is obtained by a horizontal displacement of the points on the optimal \( y_0 \) (rather than vertical displacements.)

Since we have \( \dot{x} = 1 + \alpha \dot{\lambda}(t) \) it follows that \( \dot{x} > 0 \) if \( \alpha \) is sufficiently close to zero. Then \( y_\alpha \) is the graph of a function \( y = y_\alpha(x) \) with derivative

\[ y'_\alpha(x) = \frac{\dot{\varphi}(t)}{\dot{x}(t)} = \frac{\dot{\varphi}(t)}{1 + \alpha \dot{\lambda}(t)} \]

Since \( \lambda(a) = \lambda(b) = 0 \), these functions are admissible for our problem and we have

\[ I[y_\alpha] = \int_a^b F(t + \alpha \lambda(t), \varphi(t), \frac{\dot{\varphi}(t)}{1 + \alpha \dot{\lambda}(t)}) \, dt \]

If we set \( i(\alpha) = I[y_\alpha] \), then \( i(0) \leq i(\alpha) \) since \( y_0 = \varphi \). Thus \( i'(0) = 0 \) as usual.
Now,
\[ i'(\lambda) = \int_a^b F_x(t+\lambda, \lambda, \frac{\Phi}{1+\lambda}) \lambda (1+\lambda^2) + \]
\[ + F_y(t+\lambda, \lambda, \frac{\Phi}{1+\lambda}) - \frac{\Phi}{(1+\lambda)^2} (1+\lambda^2) + \]
\[ + F(t+\lambda, \lambda, \frac{\Phi}{1+\lambda}) \lambda \int_a^b dt \]

hence
\[ b \]
\[ 0 = i'(0) = \int_a^b F_x(t, \Phi, \Phi) \lambda \]
\[ + [F(t, \Phi, \Phi) - \Phi F_y(t, \Phi, \Phi)] \lambda \int_a^b dt \]
\[ b \]
\[ = \int_a^b F_x \lambda + [F - \Phi F_y] \lambda \int_a^b dt \]

for all \( \lambda \in D_1 \) with \( \lambda(a) = \lambda(b) = 0 \). It follows from the corollary to du Bois-Reymond’s lemma that

\( (\star) \quad F - \Phi F_y = \int_a^\infty F_x(\xi) d\xi + C \)

for some constant \( C \). Here we have used that \( x = t \) such that \( \Phi(t) = \Phi(x) \) and \( \Phi(t) = \Phi'(x) \). Our result follows from (\star) because \( F_x(\xi) := F_x(\xi, \Phi(\xi), \Phi'(\xi)) \) is integrable, even at corners, hence \( F - \Phi F_y \) (and \( \int_a^b \)).
Legendre's Necessary Condition

We consider the standard problem:

\[
\begin{align*}
(P) \quad \min \ J[y] &= \int_a^b F(x, y, y') \, dx, \\
\text{Subject to } y \in D_f \text{ and } y(a) &= a, \quad y(b) = b.
\end{align*}
\]

Assuming that $\phi$ solves $(P)$ we want to prove Legendre's necessary condition

Thus $F_y \phi', \geq 0$ for all $x \in [a, b]$. *)

We assume, for simplicity, that $\phi \in C^2$.

This result essentially follows from the lemma proved below. To see why, we need some preparations: The idea is to consider $j(\varepsilon) = J[\phi + \varepsilon \eta]$ where $\eta$ is any admissible variation ($\eta \in D_f, \eta(a) = \eta(b) = 0$).

Since $j$ is (twice) differentiable and achieves its minimum at $\varepsilon = 0$ we have

\[
j''(0) = 0 \quad \text{and} \quad j''''(0) \geq 0.
\]

The first of these conditions lead us to Euler's eq. via the formula

\[
j'(0) = \int_a^b \left[ F_y \eta + F_{y'} \eta' \right] \, dx.
\]

The second (and new) condition tells us that

\[
0 \leq j''''(0) = \int_a^b \left[ F_{yy} \eta^2 + 2 F_y \eta \eta' + F_{y'y'} \eta' \right] \, dx
\]

for all admissible variations $\eta$. Since $2\eta \eta' = \frac{d}{dx}(\eta^2)$ we may integrate the middle term by parts to obtain:
\[ 0 \leq \int_a^b \left\{ \Phi \Phi' \phi'^2 + \left( \Phi \Phi - \frac{d}{dx} \Phi \Phi' \right) \phi'^2 \right\} dx \]

\[ = \int_a^b P(x) \phi'^2 + Q(x) \phi'^2 dx, \]

where \( P(x) = \Phi \Phi' \) and \( Q(x) = \Phi \Phi - \frac{d}{dx} \Phi \Phi' \).

**Lemma** If \( P \) and \( Q \) are (piecewise) continuous functions defined on \([a,b]\) and the quadratic functional

\[ I[h] = \int_a^b P(x) h'^2 + Q(x) h'^2 dx \]

is non-negative \((I[h] \geq 0)\) for all \( h \in \mathbb{D}_t \) with \( \Phi(a) = 0, \Phi(b) = 0 \), then \( P(x) \geq 0 \) for all \( x \in [a,b] \). (The proof in the book uses variations \( h \in C_1 \).

**Proof** We assume that the conclusion is false and show that this will lead to a contradiction. Suppose \( P(x_0) < 0 \) for some \( x_0 \in [a,b] \). Since \( P \) is (piecewise) continuous we may assume \( x_0 \in (a,b) \), and that there exists a number \( \delta > 0 \) such that \( P(x) < \frac{1}{2} P(x_0) \) on \([x_0 - \delta, x_0 + \delta]\).

For each \( \varepsilon > 0 \), \( 0 < \varepsilon \leq \delta \), we construct \( h_\varepsilon \in \mathbb{D}_t \) as in the figure:

\[ h_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & x_0 - \varepsilon \leq x < x_0, \\ -\frac{1}{\varepsilon}, & x_0 < x < x_0 + \varepsilon, \\ 0, & \text{otherwise}. \end{cases} \]
Now,

\[ I[h^\varepsilon] = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} P(x) h^\varepsilon_1^2 + Q(x) h^\varepsilon_2^2 \, dx \]

\[ \leq \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \frac{1}{2} P(x_0) \cdot \frac{1}{\varepsilon^2} + M h^\varepsilon_2^2 \, dx, \quad M = \max_{a \leq x \leq b} Q(x) \]

\[ = \left\{ \frac{1}{\varepsilon} P(x_0) + \frac{2}{3} M \varepsilon \right\} < 0, \]

if we choose \( \varepsilon > 0 \) sufficiently small. This contradicts the assumption that \( I[h] \geq 0 \) for all admissible \( h \).

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**Example**

We call upon an old acquaintance:

\[ \begin{cases} 
\min \int_{-1}^{1} (y^1 - 1)^2 (y^1 + 1)^2 \, dx, \\
\text{Subj. to} \quad y(-1) = 1, \quad y(1) = -1.
\end{cases} \]

The Lagrangean \( F = (y^1 - 1)^2 (y^1 + 1)^2 \) depends only on \( y^1 \), so the extremals are straight lines, so

\[ y(x) = x, \quad -1 \leq x \leq 1 \]

is the only admissible extremal. If we consider broken extremals, then every straight line segment of the broken extremal must have slope \( \pm 1 \). The broken extremal

\[ \phi(x) = |x|, \quad -1 \leq x \leq 1 \]

minimizes our functional.

Now, \( F = (y^1 - 1)^2 = y'^1 - 2y^1 + 1 \) so \( Fy'y' = 4(3y^1 - 1) \) so

\[ Fy'y' = 4(3y^1 - 1) = -4 < 0, \quad \text{not a minimizer.} \]

\[ F\phi'\phi' = 4(3\phi^1 - 1) = 4(3-1) = 8 > 0, \]

fulfills Legendre's condition.