Calculus of variations

Lecture 27

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The fundamental sufficiency condition.

References

M. Kot Ch. 12, pp. 255-267.

Newerton- Gibbons Lectures 12, 13
pp. 91-110.
An example (Shortest path)

Consider minimization of the length functional

\[ L[y] = \int_a^b \sqrt{1 + y'^2} \, dx \]

subject to \( y(a) = \alpha \) and \( y(b) = \beta \).

After translation we may assume that \( a = 0, \alpha = 0 \).

The extrema of \( L \) are straight lines, and the admissible extremal (i.e., the one satisfying \( y(0) = 0 \) and \( y(b) = \beta \)) is

\[ y_*(x) = \frac{\beta}{b} x, \quad 0 \leq x \leq b. \]

Our goal is to prove that \( y_* \) minimizes \( L \).

We introduce the new help-functional

\[ N[y] = \int_a^b \frac{x + y^2 y'}{\sqrt{x^2 + y^2}} \, dx \]

and observe that since \( \frac{d}{dx} \sqrt{x^2 + y^2} = \text{integral} \),

\[ N[y] = \left[ \sqrt{x^2 + y(x)^2} \right]_a^b \]

\[ = \sqrt{b^2 + \beta^2} - \sqrt{a^2 + \alpha^2} = \sqrt{b^2 + \beta^2}. \]

Thus \( N \) depends only on the end points of \( y \), and not on the function values in between. (In fact \( \frac{x + yy'}{\sqrt{x^2 + y^2}} \) is a null-adjugate, it.)
It follows that the minimizer of
\[ L[y] - N[y] \]
is the same as the minimizer of \( L \), when both problems are subject to the same boundary conditions \( y(0) = 0 \) and \( y(\beta) = \beta \).

Now,
\[
L[y_\ast] - N[y_\ast] = \int_0^\beta \sqrt{1 + (\beta \lambda)^2} \, dx - \int_0^\beta \sqrt{x^2 + \beta^2} \, dx
\]
\[
= \sqrt{b^2 + \beta^2} - \sqrt{b^2 + \beta^2} = 0
\]
and
\[
L[y] - N[y] \geq 0
\]
for any admissible \( y \); in fact the lagrangian of \( L - N \) satisfies
\[
\sqrt{1 + y'^2} - \frac{x + y'y'}{\sqrt{x^2 + y^2}} \geq \sqrt{1 + y'^2} - \frac{\sqrt{x^2 + y^2} - \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} = 0
\]
by the Cauchy-Schwarz inequality. Therefore \( y_\ast \) minimizes \( L - N \) and consequently also \( L \). (Equality holds in the above ineq. if and only if \( y = kx \) for some \( k \in \mathbb{R} \). This proves uniqueness of the minimizer.)
Fields and Hilbert's Invariant Integral

Tools (a) We are going to use Green's formula:

$$\int_D P\, dx + Q\, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx\, dy$$

($P, Q$) is a $C^1$ vector field defined in a neighbourhood of $\overline{D} = D \cup \partial D$.

(b) Application: If the differential form $\omega = P\, dx + Q\, dy$ is closed $[d\omega = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \, dx\, dy = 0]$ then the integral

$$I = \int_D P\, dx + Q\, dy$$

is independent of the path $\gamma$ connecting $A$ to $B$; $I[\gamma] = I[\gamma']$.

(c) Proof of claim:

$$I[\gamma'] - I[\gamma] = \int_D \omega - \int_D \omega = \int_D \omega = \int_D d\omega = 0$$

(d) Special Case: Suppose $\gamma$ is the graph of a function $y = y(x), a \leq x \leq b$. Then

$$\int_D P\, dx + Q\, dy = \int_{\gamma} dy = y' \, dx = \int_a^b \left( P + y' Q \right) \, dx$$.
Fields of Extremals & Direction Fields

Extremals of the functional

\[ J[y] = \int_a^b F(x, y, y') \, dx \]

are the solutions of Euler's equation

\[ F_y - \frac{d}{dx} F_{y'} = 0. \]  

This is a second-order ODE giving, in general, a two-parameter family of extremals:

\[ y = \phi(x; k, l), \]

parameter names \( k, l \) being the same as in the book.

A one-parameter (sub-)family of extremals on an open domain \( R \subseteq \mathbb{R}^2 \),

\[ y = \phi(x; \kappa) \]

is called a field of extremals for \( J \) if the solution curves cover an open portion \( R \) of the \( xy \)-plane such that

Through each point \((x_0, y_0) \in R\) there passes exactly one extremal: For unique \( \kappa \)

\[ y_0 = \phi(x_0; \kappa). \]
Remark (i) The above condition means that for each $(x_0, y_0) \in \mathbb{R}$ there exists a unique parameter $\kappa = \kappa(x_0, y_0)$ such that the extremal $y = \phi(x; \kappa(x_0, y_0))$ satisfies

$$y_0 = \phi(x_0; \kappa(x_0, y_0)).$$

(ii) The extremals $y = \phi(x; \kappa)$ cannot intersect in $\mathbb{R}$. For suppose $y = \phi(x; \kappa_1)$ and $y = \phi(x; \kappa_2)$ intersected at $(x_0, y_0) \in \mathbb{R}$, then the definition of a field of extremals would be contradicted! [This is related to Jacobi's eqn. and the notion of conjugate points.]

The direction field associated with the field $y = \phi(x; \kappa)$ is the scalar func. $\rho : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho(x, y) = \text{slope at } x_0 \text{ of the extremal } y = \phi(x; \kappa) \text{ passing through } (x_0, y_0) \in \mathbb{R}.$$  

Since $y' = \phi_x(x_0, \kappa(x_0, y_0))$, we get

$$\rho(x_0, y_0) = \phi_x(x_0, \kappa(x_0, y_0)).$$
Remark. We know that the extremals of a functional \( I = \int_{a}^{b} F(x, y, y') \, dx \) in general is a two-parameter family of functions, because Euler's equation \( F_y - \frac{d}{dx} F_y = 0 \) is a second-order ODE. It is therefore natural to ask, the other way around, whether the extremals in a field \( \phi = \phi(x, \kappa) \), which form a one-parameter family of functions, are the solutions of a first-order ODE.

The answer is affirmative; in fact, this is precisely what the direction field does:

\[(\star) \quad \phi'(x) = P(x, \phi(x))\]

where \( \phi(x) = \phi(x, \kappa) \) is any member of the field.
An additional tool: Let \( \phi \) one of the extremals in a field \( \phi = \phi(x, y) \) and denote by \( \rho = \rho(x, y) \) the corresponding direction field.

For any function \( h = h(x, y) \) def. on \( \mathbb{R}^2 \), the domain on which the field is defined, such that \( h \in C^1 \) we have

\[
\frac{d}{dx} h(x, \phi(x)) = h_x(x, \phi(x)) + \phi'(x) h_y(x, \phi(x))
\]

(using \( (x) \))

(in Remark) = \[ h_x(x, \phi(x)) + \rho(x, \phi(x)) h_y(x, \phi(x)) \]

= \[ h_x(x, y) + \rho(x, y) h_y(x, y) \bigg|_{y = \phi(x)} \]

= \[ \left( \frac{\partial}{\partial x} + \rho(x, y) \frac{\partial}{\partial y} \right) h(x, y) \bigg|_{y = \phi(x)} \]

We summarize:

\( (\star \star) \quad \frac{d}{dx} h(x, \phi(x)) = \left( \frac{\partial}{\partial x} + \rho(x, y) \frac{\partial}{\partial y} \right) h(x, y) \bigg|_{y = \phi(x)} \)

which will be used as an essential part of the following derivation.
Example (p. 96) Westerton - Gibbons

\[ J[y] = \int_1^3 \frac{1}{2} y'^2 + y'y + y' + y \, dx \]

(P) Minimize \( J \) subject to \( y(1) = 0, \ y(3) = 4 \).

We have the Lagrange function

\[ F(x, y, y') = \frac{1}{2} y'^2 + y'y + y' + y, \] so that

\[ \frac{\partial F}{\partial y} = y' + 1, \ \frac{\partial F}{\partial y'} = y' + y + 1. \] Euler's eqn. becomes

\[ y'' = 1 \]

(*) \[ y = \phi(x; k, l) = \frac{1}{2} x^2 + k x + l; \quad k, l \in \mathbb{R} \] a two-parameter family of extremals.

The admissible extremal is

\[ \bar{y} = \phi_0(x) = \phi(x; 0, -\frac{1}{2}) = \frac{1}{2} x^2 - \frac{1}{2}. \] This will be used later.

Fields of extremals can be constructed in many different ways using (*) Here are two particularly simple choices

**Subfamily I**

\[ y = \phi_2(x; k) = \phi(x; k, 0). \] (A family of parabolas through the origin, \( y = \frac{1}{2}(x+k)^2 - \frac{1}{2}k^2 \))

\[ \phi_{2,1}(x; k) = x + k \]

So

\[ \phi_2(x, y) = \frac{y}{x} + \frac{x}{2}, \quad x > 0. \]

Field in \( R = \{(x, y) : \ 1 < x < 3 \}. \)
Subfamily II \[ y = \phi_2(x; \ell) = \phi(x; 0, \ell) = \frac{1}{2} x^2 + \ell \]

(A family of vertically displaced parabolas)

\[ \phi_2, x(x; \ell) = x \]

hence

\[ \rho_2(x, y) = x. \]

Notice that the admissible extremal \[ \phi_0(x) = \frac{1}{2} x^2 - \frac{1}{2} \] is embedded in the second field of extremals, but not in the first! This will be used later.

Hilbert's Invariant Integral

For a curve \( y = y(x) \) define

\[ K[y] = \int_{a}^{b} \left\{ F(x, y, \sigma) - \sigma F_y(x, y, \sigma) \right\} + y'(x) F_y(x, y, \sigma) \, dx \]

where \( \sigma = \sigma(x, y) \) is a direction field. If we set

\[ P(x, y) = F(x, y, \sigma) - \sigma F_y(x, y, \sigma) = -H(x, y, \sigma) \]

\[ Q(x, y) = F_y(x, y, \sigma) \]

then

\[ K[y] = \int_{\gamma} P \, dx + Q \, dy. \]
$K[xy]$ is called \textit{Hilbert's invariant integral}. The name stems from the fact that $K[xy]$ has the same value for all curves connecting the same two end points.

To prove this we have to verify that $\partial Q/\partial x - \partial P/\partial y = 0$ in the region $R$ where the field is defined. ($R$ is assumed simply connected.)

\begin{align*}
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{\partial}{\partial x} F_y'(x,y,x) - \frac{\partial}{\partial y} [F(x,y,x) - gF_y'(x,y,x)] \\
&= \left( \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right) F_y'(x,y,x) + g_y F_y'(x,y,x) \\
&\quad - F_y(x,y,x) - F_y'(x,y,x)y \\
&= \left( \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right) F_y'(x,y,x) - F_y(x,y,x)
\end{align*}

Using that $(\partial x + g \partial y)F_y'(x,y,x) = \frac{d}{dx} F_y(x,y,y')$ along extremals, we find:

\begin{align*}
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{d}{dx} F_y'(x,y,y') - F_y(x,y,y') \\
&= 0, \quad (\text{Euler's eqn.} !)
\end{align*}

as wanted.
Example. We consider again the functional \( J[y] = \int_1^3 \left\{ \frac{1}{2} y'^2 + y'y + y' + y \right\} \, dx \).

We have seen that
\[
\phi_1(x; k) = \frac{1}{2} x^2 + kx, \quad k \in \mathbb{R}, \quad \text{and}
\phi_2(x; t) = \frac{1}{2} x^2 + t, \quad t \in \mathbb{R},
\]

are two fields of extremals.

Field 1: Here the direction field is
\[
\rho = \rho(x, y) = \frac{y}{x} + \frac{x}{2},
\]
and Hilbert's integral becomes
\[
K[y] = K[y] = \int_1^3 \{ f(x, y, \rho) + (y' - \rho)y' \} \, dx = \frac{1}{2} x^2 + gy + g + y + (y' - \rho)(g + y + t) \, dx
\]
\[
= \int_1^3 \left\{ \left( \frac{y}{x} + \frac{x}{2} + y + t \right)y' - \frac{1}{2} \left( \frac{y}{x} - \frac{x}{2} \right)^2 \right\} \, dx
\]
\[
= \int_1^3 \frac{d}{dx} \left( \frac{y^2}{x} + xy + y^2 + 2y - \frac{x^3}{12} \right) \, dx = \text{constant}.
\]

\[
U(x, y) = U(3, y(3)) - U(1, y(1)),
\]
which is clearly independent of \( y = y(x) \). A similar result holds for \( \phi_2 \), the second field.

7
Sufficient Conditions

Consider the standard problem; to minimize

\[ J[y] = \int_a^b F(x, y, y') \, dx \quad (P) \]

subject to \( y(a) = \alpha, \ y(b) = \beta. \)

We have proved Weierstrass' necessary condition: If \( \phi \) is a strong relative minimum of \( (P) \) then

\[ (W) \quad \vartheta(x, \phi(x), \phi'(x), \omega) \geq 0, \forall \omega \in \mathbb{R}, \]

where \( \vartheta(x, y, y', \omega) \) is Weierstrass' excess function defined by

\[ \vartheta(x, y, y', \omega) = F(x, y, \omega) - F(x, y, y') - (\omega - y') F_y(x, y, y'). \]

The Weierstrass' condition \( (W) \) is also an essential ingredient in the following sufficient condition which ensures that an admissible extremal \( y = \phi(x) \) is a strong relative minimum (solution) of \( (P) \).

We assume \( y = \phi(x) \) is an admissible extremal for \( P \) and that \( \phi_* \) is embedded in a field of extremals \( y = \phi(x, \kappa) \), [so that \( \phi_* = \phi(\cdot, \kappa_*) \) for some \( \kappa_* \)], and let \( \mathcal{P} = \mathcal{P}(x, y) \) be the direction field.
We want to show that
\[ J[y] - J[\phi_*] \geq 0 \]
for all \( y \) in a (strong) neighborhood of \( \phi_* \).

With the notation \( K[y] = K[y] \) if \( y : y = y(x) \), we find:

\[
K[\phi_*] = \int_a^b F(x, \phi_*(x), \phi'_*(x)) + \phi'_*(x) F_y(x, \phi_*(x), g) \, dx
\]

\[(\star) = \int_a^b F(x, \phi_*(x), \phi'_*(x)) + 0 \, dx
\]

= \( J[\phi_*] \) because \( \phi'_*(x) = g(x, \phi_*(x)) \).

Now:

\[
J[y] - J[\phi_*] = J[y] - K[\phi_*]
\]

\[(\star) \]

= \( J[y] - K[y] = \)

\[
b \int_a^b F(x, y, y') - F(x, y, g) - (y' - g) F_y(x, y, g) \, dx
\]

= \( \int_a^b \mathcal{G}(x, y, g(x, y), y') \, dx \)

so \( J[y] - J[\phi_*] \geq 0 \) if \( \mathcal{G}(x, y, g(x, y), w) \geq 0 \) for all \( w \in TR \).
We have proved

**Theorem.** Let \( y^* : y = \phi_*(x) \) be an admissible extremal imbedded in a field of extremals with direction field \( p = p(x, y) \). If \( \mathcal{E}(x,y,p(x,y),\omega) \geq 0 \) for all (feasible) \( \text{WTR} \), then \( \phi_* \) is a strong relative minimum of \( J[y] \) (through the points \((a, \alpha)\) and \((b, \beta)\)).

Remark. The condition \( \mathcal{E}(x,y,p(x,y),\omega) \geq 0 \) for all \( \text{WTR} \) means that Weierstrass' condition holds along all the extremals in the (chosen) field ... not only along the proposed minimizing extremal \( \phi_*(x) \).

**Example.** The problem of minimizing

\[
J[y] = \int x^{3/2} y'^2 + y^2 + y' + y \ dx,
\]

subject to

\( y(1) = 0 \) and \( y(3) = 4 \), has the admissible extremal \( \phi_*(x) = \frac{1}{2} x^2 - \frac{1}{4} \). Now, \( \phi_* \) is imbedded in a field of extremals, namely

\[
\phi_2(x; \ell) = \frac{1}{2} x^2 + \ell, \text{ } \ell \in \mathbb{R}.
\]

The direction field is, as we have seen, \( p(x,y) = x \).

\( \mathcal{E}(x,y,y',\omega) = \ldots = \frac{1}{2}(\omega - y')^2 \), so Weierstrass' condition is satisfied

\[
\mathcal{E}(x,y,p(x,y),\omega) = \frac{1}{2}(\omega - x)^2 \geq 0, \text{ } \forall \text{WTR},
\]

hence \( \phi_* \) is a strong relative minimum.
Observation: Weierstrass' excess function is the remainder in the first-order Taylor expansion of \( F(x,y,y') \) around \( y' \):

\[
\varepsilon(x,y,y',\omega) = F(x,y,\omega) - F(x,y,y') - (\omega - y') F_{y'}(x,y,y')
\]

\[
= \frac{1}{2} F_{yy'}(x,y,\theta x + (1-\theta)y' + \theta \omega) \cdot (\omega - y')^2,
\]

where \( \theta = \theta(x,y) \) satisfies \( 0 \leq \theta \leq 1 \). Thus, if \( F_{yy'}(x,y,y') \geq 0 \) for all \( (x,y,y') \), then so is \( \varepsilon \).

Application (The Brachistochrone, again.)

The Lagrange function of the brachistochrone is \( F = \sqrt{1+y'^2}/y \), so that

\[
F_{yy'}(x,y,y') = \frac{1}{\sqrt{y} (1+y'^2)^{3/2}} > 0
\]

Now, the extremals are cycloids, and the extremals through the origin \((x,y) = (0,0)\) form a field, in which the admissible extremal through \((-b, \beta)\) is embedded. Since \( \varepsilon > 0 \), cycloids are minimizers (global).