Calculus of Variations — Lecture 1

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Exercises (with some hints below) Remember that (most of) you haven't got any problem specific tools yet, they will be presented in the following two lectures. Instead, use your imagination, show some ingenuity and use your calculus skills!

1. Let J^* denote the minimum value of the functional

$$J[y] = \int_0^1 y(x)^2 y'(x)^2 \, dx,$$

when minimized over differentiable functions $y : [0,1] \to \mathbf{R}$ satisfying the end-point conditions $y(0) = 0$ and $y(1) = 1$.

a) Determine an *upper bound* on J^* by restricting the investigation to (trial-)functions of the type $y_s(x) = x^s$ where $s > 1/4$ (why this condition?). That is, by minimizing the function $f(s) := J[y_s]$ over the interval $(1/4, +\infty)$.
(Adapted from M. Mesterton-Gibbons, Exercise 1.1.)

b) Guess the function which solves the variational problem. Can you verify your guess?

2. a) Find the function $y_0 \in C^1([0,1])$ satisfying $y_0(0) = 1$ and $y_0(1) = 3/2$ which minimizes the functional

$$J(y) = \int_0^1 (y'(x) - x)^2 \, dx.$$

(How can you be sure you've got the right solution?)

b) Same problem but this time for functions satisfying $y_0(0) = 0$ and $y_0(1) = 1$.

3. Try to guess the solution of the problem of minimizing the functional

$$J[y] = \int_0^2 y(x)^2 (1 - y'(x))^2 \, dx$$

subject to $y(0) = 0$ and $y(2) = 1$. Which function class does your guess belong to? How do you know that the function you guessed is a minimizer?

4. Show that the straight line $\phi(x) = \alpha + (\beta - \alpha)(x - a)/(b - a)$ minimizes the curve length functional

$$L[y] = \int_a^b f(y'(x)) \, dx, \quad f(p) = \sqrt{1 + p^2},$$

subject to $y(a) = \alpha$ and $y(b) = \beta$, by establishing the inequality $L[y] - L[\phi] \geq 0$ for all admissible y. Do this by writing $y(x) = \phi(x) + h(x)$, with $h(a) = h(b) = 0$, and make a first-order Taylor expansion of $f(\phi + h)$ with remainder in Lagrange's form. (This is another example of a *direct verification*.)

Hint for 1 b) Suppose you have guessed that the function $\phi(x)$ is a minimizer of the problem in 1. One way to verify optimality would be to compare the value $J[\phi]$ to a lower bound on $J[y]$ obtained from the Cauchy-Schwarz (integral) inequality: If f and g are functions defined and integrable on the interval $[a, b]$, then

$$\int_a^b f(x)g(x)\,dx \leq \left(\int_a^b f(x)^2\,dx\right)^{1/2} \left(\int_a^b g(x)^2\,dx\right)^{1/2}$$

with equality if and only if there exists constants $A, B \geq 0$, not both zero, such that $Af(x) + Bg(x) = 0$. (Notice that there are several more ways to solve this problem.)

Hint for 4. Suppose we want to verify that a function $\phi(x)$ minimizes $L[y]$, then we need to show that $L[\phi] \leq L[y]$ for all admissible functions y. Any admissible y can be expressed in the form $y(x) = \phi(x) + h(x)$ where h is continuously differentiable and satisfies $h(a) = 0, h(b) = 0$ (why?). Taylor's theorem states that

$$f(\phi'(x) + h'(x)) = f(\phi'(x)) + f'(\phi'(x))h'(x) + \frac{1}{2} f''(\phi'(x) + \theta(x)h'(x))h'(x)^2$$

for some continuous function $\theta(x)$ satisfying $0 < \theta(x) < 1$ for all $a < x < b$. The second derivative $f''(z)$ is positive for all z.