Calculus of Variations — Lecture 3

31 January 2019, Niels Chr Overgaard

Exercises

1. Minimize the functional

$$J[y] = \int_1^2 \frac{1}{4} y'(x)^4 \, dx$$

subject to the end point conditions $y(1) = 0$ and $y(2) = 3$. Use a direct verification to show that the minimum is achieved.

2. Show that the problem of minimizing the functional

$$J[y] = \int_0^2 y(x)^2 (1 - y'(x))^2 \, dx$$

subject to the end point conditions $y(0) = 0, y(2) = 1$ does not have any admissible extremals. Find by inspection a broken extremal that minimizes J given the end point conditions above.

3a. Determine the admissible extremals of the functional

$$I[x] = \int_1^2 t^3 \dot{x}(t)^2 \, dt$$

subject to the end point conditions $x(1) = 0$ and $x(2) = 3$.

3b. Determine the admissible extremals of the functional

$$I[x] = \int_{\frac{1}{2}}^1 \frac{\dot{x}(s)^2}{s} \, ds$$

subject to the end point conditions $x(1/2) = 3$ and $x(1) = 0$.

4. Confirm by direct verification that the minimizers determined in 3a and 3b are minimizers of the respective functionals. What is the minimum value in each case?

(The problems 2, 3a,b and 4 are adapted from Mesterton-Gibbons book.)