Calculus of Variations — Lecture 4/5

18 February 2019, Niels Chr Overgaard

Exercises

1. Let $m = m(x)$ be a continuous function defined on $[a, b]$ and define

$$J[y] = \int_a^b \frac{1}{2}(y'(x) - m(x))^2 \, dx$$

Determine the extremals of this functional (Using, for instance, Euler's equation in integrated form by Du Bois Reymond). Find an example (i.e., a choice of a, b, m and possibly end-point conditions) which shows that the extremals may belong to C^1 but not to C^2.

2. During most of the 19th century many mathematicians seemed to believe that any functional J with a non-negative integrand must, self-evidently, possess an admissible extremal y_0 which minimizes the functional. Around 1870 Karl Weierstrass, then professor in Berlin, challenged this belief with the following problem: Minimize the integral

$$J[y] = \int_{-1}^{1} x^2 y'(x)^2 \, dx$$

over the set of functions $y \in C^1$ which satisfies the end point conditions $y(-1) = -1$ and $y(1) = 1$.

a) Verify that J is bounded below by zero, i.e., $J[y] \geq 0$.

b) Compute (as Weierstrass did) the values $J[y_\epsilon]$ where

$$y_\epsilon(x) = \frac{\arctan(x/\epsilon)}{\arctan(1/\epsilon)}, \quad \epsilon > 0.$$

Determine the greatest lower bound on $J[y]$ (denoted $\inf_y J[y]$) when y ranges over the set of admissible functions.

c) Does there exist an admissible function y_0 such that $J[y_0] = \inf_y J[y]$?

d) The sequence of functions $\{y_\epsilon\}_{\epsilon>0}$ in b) is an example of a so-called minimizing sequence for J. Verify that the following construction also gives a minimizing sequence: Take any non-decreasing function $\phi \in C^1(\mathbf{R})$ which satisfies $\phi(x) = -1$ for $x \leq -1$ and $\phi(x) = 1$ for $x \geq 1$ (draw a figure!) and set

$$\phi_\epsilon(x) = \phi(x/\epsilon), \quad 0 < \epsilon \leq 1.$$

3. Derive Euler's equations for the problem of minimizing curve length

$$L[\mathbf{r}] = \int_0^1 \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} \, dt$$

for curves $\mathbf{r} : [0,1] \to \mathbf{R}^3$, $\mathbf{r}(t) = (x(t), y(t), z(t))$ lying on the unit sphere S^2 : $x^2 + y^2 + z^2 = 1$ and connecting two given (distinct) points P_0 and P_1 on the sphere. (Introduce a suitable parametrization for the unit sphere and transform (pull back) the minimization problem to the parameter domain. ... If you haven't tried it before, you'll have to think a little!...But you'll learn a lot.)

4. Show that the functional defined in Problem 3 (and any other curve length functional, for that matter) satisfies

$$L[\mathbf{r}] = L[\mathbf{r} \circ \varphi],$$

where $\varphi : [0,1] \to [0,1]$ is any strictly increasing continuously differentiable function satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. The curve $\mathbf{r} \circ \varphi$ is just a reparametrized version of $\mathbf{r}$. What is the geometric significance of this property of the length functional L?

5. The following problem connects problems 3 and 4 above. Suppose $\mathbf{r}_0 : [0,1] \to S^2$ is the shortest path on the sphere connecting P_0 to P_1, i.e., a solution to our variational problem. Show first that a) we may assume, without loss of generality, that the path is traversed with constant speed. Then show that b) if the path $\mathbf{r}_0$ is considered as the motion of a particle of mass $m = 1/2$, then it is a minimizer of the action integral,

$$A[\mathbf{r}] = \int_0^1 \frac{1}{2} m \dot{\mathbf{r}}(t)^2 \, dt = \frac{1}{2} m \int_0^1 \dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 \, dt.$$

(Hint: don't forget about Cauchy-Schwarz.) This observation plays a central role in differential geometry because the action functional has a simpler Euler-Lagrange equation than the path length functional.