Solutions

1. Setting \( y = x' \) we obtain the equivalent system

\[
\begin{cases}
    x' = y \\
y' = -2xy - x^3 + x.
\end{cases}
\]

Any fixed point \((x, y)\) must satisfy

\[
\begin{cases}
y = 0 \\
-2xy - x^3 + x = 0.
\end{cases}
\]

Substituting \( y = 0 \) into the second equation yields \( x^3 - x = x(x - 1)(x + 1) = 0 \). The fixed points are therefore \((0, 0)\), \((1, 0)\) and \((-1, 0)\). The Jacobi matrix is

\[ f'(x, y) = \begin{pmatrix} 0 & 1 \\ -3x^2 - 2y + 1 & -2x \end{pmatrix} \]

We have

\[ f'(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

with eigenvalues \( \lambda = \pm 1 \),

\[ f'(1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \]

with eigenvalues \( \lambda = -1 \pm i \) and

\[ f'(-1, 0) = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \]

with eigenvalues \( \lambda = 1 \pm i \). Hence, \((1, 0)\) is asymptotically stable while the other two fixed points are unstable.

2. We have \( x^2 \geq \pi^2 > 3^2 = 9 \) on the interval \([\pi, 2\pi]\). The equation \( y'' + 9y = 0 \) has the solution \( y(x) = \sin(3x) \), with zeros at \( \pi \), \( 4\pi/3 \), \( 5\pi/3 \) and \( 2\pi \). By Sturm’s comparison theorem it follows that any solution of \( y'' + x^2y = 0 \) has at least three zeros in \( (\pi, 2\pi) \) (at least one in each of the subintervals \( (\pi, 4\pi/3) \), \( (4\pi/3, 5\pi/3) \) and \( (5\pi/3, 2\pi) \)).

3. a) The eigenvalues of a Sturm-Liouville problem are real.

If \( \lambda = -k^2 < 0 \), we obtain that \( y(x) = a \cosh(kx) + b \sinh(kx) \). The boundary conditions give \( bk = 0 \) and \( ak \sinh(k\pi) + bk \cosh(k\pi) = 0 \), so \( a = b = 0 \).
If $\lambda = 0$, we obtain that $y(x) = ax + b$, which satisfies the equation iff $a = 0$. Hence $\lambda_0 = 0$ is an eigenvalue with normalized eigenfunction $u_0(x) = \sqrt{\frac{2}{\pi}}$.

If $\lambda = k^2 > 0$, we obtain that $y(x) = a\cos(kx) + b\sin(kx)$. The boundary conditions are satisfied iff $bk = 0$ and $ak\sin(k\pi) = 0$. This gives a non-trivial solution iff $k$ is a non-zero integer. Hence $\lambda_n = n^2$, $n = 1, 2, 3, \ldots$ is an eigenvalue with normalized eigenfunction $u_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx)$.

b) We have

$$f(x) = \sum_{n=0}^{\infty} c_n u_n,$$

where $c_n = \langle u_n, f \rangle$. For $n = 0$, we obtain that

$$c_0 = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \sin(x) \, dx = \frac{2}{\sqrt{\pi}}.$$

For $n \geq 1$, we obtain that

$$c_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \sin(x) \cos(nx) \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \frac{1}{2} \left( \sin((1 + n)x) + \sin((1 - n)x) \right) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi} \left[ \cos((1 + n)x) - \cos((1 - n)x) \right] \, dx$$

$$= \begin{cases} 0, & n \text{ odd,} \\ \sqrt{\frac{2}{\pi}} \frac{2}{1 - n^2}, & n \text{ even.} \end{cases}$$

Hence,

$$\sin x = \frac{2}{\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{4}{1 - 4k^2} \cos(2kx).$$

The series converges in $L^2$ and uniformly on $[0, \pi]$ (Weierstrass’ M-test).

4. All points on the $x$-axis are fixed points and the positive $y$-axis is an orbit. Since different orbits cannot intersect, it follows that the open first quadrant is invariant. Since $x'(t) < 0$ in the open first quadrant, it follows that we can write $y$ as a function of $x$ along a solution. We have $dy/dx = y'/x' = -1 + 1/x$. Each orbit is therefore contained in a curve $y = -x + \ln x + C$ for some constant $C > 1$. The curve intersects the positive $x$-axis in two different points $x_1$ and $x_2$, with $x_1 < x_2$. We have $\lim_{t \to \infty}(x(t), y(t)) = (x_1, 0)$ and $\lim_{t \to -\infty}(x(t), y(t)) = (x_2, 0)$. 
5. Suppose that the solution $x(t)$ stays within the ball of radius $R > 0$ for all times. Then $\frac{d}{dt}E(x(t)) \geq E_0$, where $E_0 = \min_{|x| \leq R} \dot{E}_f(x) > 0$. Hence, $E(x(t)) \geq E_0 t + E(x(0))$. On the other hand $E(x(t)) \leq \max_{|x| \leq R} E(x)$. This is a contradiction.