Boundary value problems

Initial value problem (IVP): find a solution $y(t)$ of an ODE with conditions given at some initial point $t = t_0$.

Ex:

$$\begin{align*}
y''(t) + 2y'(t) + 3y(t) &= \sin t, \quad t \geq t_0 \\
y(t_0) &= 1, \\
y'(t_0) &= 2.
\end{align*}$$

Boundary value problem (BVP): find a solution on some given interval $[a, b]$ with conditions at the endpoints $a$ and $b$.

Ex:

$$\begin{align*}
y''(x) + 2y'(x) + 3y(x) &= \sin x, \quad a \leq x \leq b \\
y(a) &= 1, \\
y'(b) - y(b) &= 2.
\end{align*}$$

Note: $x$ used instead of $t$, since the independent variable often has the meaning of a spatial coordinate in BVPs.

Separation of variables

Wave eq. (PDE)

$$\frac{\partial^2 u}{\partial t^2}(t, x) = c^2 \frac{\partial^2 u}{\partial x^2}(t, x),$$

$c > 0$ constant. Describes small vibrations in a homogeneous string. $u(t, x)$ = displacement of the string at position $x$ and time $t$. Assume fixed at endpoints: $u(t, 0) = u(t, 1) = 0$ (length 1).

Look for a solution of the form

$$u(t, x) = w(t)y(x).$$
Gives
\[ w''(t)y(x) = c^2w(t)y''(x) \iff \frac{w''(t)}{c^2w(t)} = \frac{y''(x)}{y(x)}. \]
LHS independent of \( x \), RHS indep. of \( t \), so both must be constant = \(-\lambda\). Gives
\[ -w'' = \lambda c^2 w(t) \]
and
\[ -y''(x) = \lambda y(x), \quad y(0) = y(1) = 0. \]
Thus,
\[ w(t) = c_1 \cos(c\sqrt{\lambda}t) + c_2 \sin(c\sqrt{\lambda}t), \]
\[ y(x) = c_3 \cos(\sqrt{\lambda}x) + c_4 \sin(\sqrt{\lambda}x). \]
Boundary condition \( y(0) = 0 \) gives
\[ c_3 = 0. \]
Boundary condition \( y(1) = 0 \) gives
\[ c_4 \sin(\sqrt{\lambda}) = 0. \]
Nontrivial solution \( \Leftrightarrow \lambda = n^2\pi^2, \ n = 1, 2, 3, \ldots \) Then,
\[ u(t, x) = (c_1 \cos(cn\pi t) + c_2 \sin(cn\pi t)) \sin(n\pi x). \]
Normally one also has initial conditions \( u(0, x) = u(x), \ \frac{\partial u}{\partial t}(0, x) = v(x) \). Can try to solve this by making an Ansatz in the form of a series
\[ u(t, x) = \sum_{n=1}^{\infty} (c_{1,n} \cos(cn\pi t) + c_{2,n} \sin(cn\pi t)) \sin(n\pi x). \]
This is formally a solution by the superposition principle. See Lemma 5.1 about convergence.
Evaluating at \( t = 0 \), we formally obtain
\[ u(x) = \sum_{n=1}^{\infty} c_{1,n} \sin(n\pi x), \]
\[ v(x) = \sum_{n=1}^{\infty} cn\pi c_{2,n} \sin(n\pi x). \]
Can ‘any’ function be written like this? Fourier analysis (special case of what we do later). The functions \( \sin(n\pi x) \) can be seen as ‘eigenvectors’ for the linear operator
\[ y \mapsto -y'' \]
with boundary conditions \( y(0) = y(1) = 0 \). Eigenvalues \( \lambda_n = n^2\pi^2, \ n = 1, 2, \ldots \)
Solvability

We know what kind of initial conditions (in particular how many) we can give for an IVP to obtain a unique solution. We now study the same question for BVPs. We restrict ourselves to 2nd order linear ODEs:

\[ a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \quad a \leq x \leq b. \]

Short form:

\[ \mathcal{L}y = f. \]

The coefficients are assumed to be continuous with \( a_2(x) \neq 0 \) on \([a, b]\).

General form of boundary conditions:

\[
\begin{align*}
 b^{a}_{11}y(a) + b^{a}_{12}y'(a) + b^{b}_{11}y(b) + b^{b}_{12}y'(b) &= c_1, \\
 b^{a}_{21}y(a) + b^{a}_{22}y'(a) + b^{b}_{21}y(b) + b^{b}_{22}y'(b) &= c_2.
\end{align*}
\]

Short form:

\[ B y = c \quad \text{or} \quad \begin{cases} B_1 y = c_1 \\ B_2 y = c_2, \end{cases} \]

where \( c = (c_1, c_2) \). We’ll see below why we need precisely 2 BCs.

Remark: don’t need to involve higher derivatives. Why?

Examples.

- **Dirichlet conditions**: \( b^{a}_{11} = b^{b}_{21} = 1, \) the rest 0. \( y(a) = c_1, \ y(b) = c_2. \)
- **Neumann conditions**: \( b^{a}_{12} = b^{b}_{22} = 1, \) the rest 0. \( y'(a) = c_1, \ y'(b) = c_2. \)
- **Periodic conditions**: \( b^{a}_{11} = b^{b}_{22} = -b^{b}_{11} = -b^{b}_{22} = 1, \) the rest 0. \( c_1 = c_2 = 0. \)

The first two are examples of separated BCs (i.e. only one endpoint per BC).

General form of BVP:

\[
\begin{align*}
 \mathcal{L}y &= f, \\
 B y &= c.
\end{align*}
\]

Recall that the solutions of \( \mathcal{L}y = 0 \) form a 2-dim. vector space. Given a basis \( \{y_1, y_2\} \), the general solution of \( \mathcal{L}y = f \) can be written

\[ y_{\alpha_1, \alpha_2} = \alpha_1 y_1 + \alpha_2 y_2 + \tilde{y}, \quad \alpha_1, \alpha_2 \in \mathbb{C}, \]

\( \tilde{y} \) particular solution of \( \mathcal{L}y = f. \) In order to solve (1) we need to choose \( \alpha_1, \alpha_2 \) s.t. \( By_{\alpha_1, \alpha_2} = c. \) Note that

\[
B(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 B y_1 + \alpha_2 B y_2 = \begin{pmatrix} B_1 y_1 & B_1 y_2 \\ B_2 y_1 & B_2 y_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.
\]

where \( c = (c_1, c_2) \). We’ll see below why we need precisely 2 BCs.
Theorem. The following are equivalent.

\begin{itemize}
\item[a)] \((1)\) has a unique solution for all \(f \in C[a, b]\) and \(c \in \mathbb{C}^2\).
\item[b)] The problem \(Ly = 0, By = c\) has a unique solution for all \(c \in \mathbb{C}^2\).
\item[c)] The problem \(Ly = 0, By = 0\) only has the trivial solution \(y(x) \equiv 0\).
\item[d)] \(\det(B_{jk} y_k) \neq 0\).
\end{itemize}

Proof. Note that \(L(y - \tilde{y}) = 0, B(y - \tilde{y}) = c - B\tilde{y}\). It follows that a) \(\iff\) b).

Any solution of b) has the form \(y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x)\). By (2) \(y\) is a solution iff
\[
\begin{pmatrix}
B_1 y_1 & B_1 y_2 \\
B_2 y_1 & B_2 y_2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}.
\]
This has a unique solution for all \(c \in \mathbb{C}^2 \iff\) the matrix \((B_{jk} y_k)\) is invertible. But this is equivalent to d). It’s also equivalent to the problem
\[
\begin{pmatrix}
B_1 y_1 & B_1 y_2 \\
B_2 y_1 & B_2 y_2
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
only having the trivial solution \((\alpha_1, \alpha_2) = (0, 0)\), which in turn is equivalent to c). \(\square\)

Example. For which \(\lambda \in \mathbb{C}\) is the problem \(y'' + \lambda y = f, y(a) = y(b) = 0\) uniquely solvable?

If \(\lambda \neq 0\), choose \(y_1(x) = e^{i\sqrt{\lambda}x}, y_2(x) = e^{-i\sqrt{\lambda}x}\) (where \(\sqrt{\lambda}\) is one of the two square roots). Since \(B_1 y = y(a)\) and \(B_2 y = y(b)\), condition c) in the theorem is
\[
\begin{vmatrix}
e^{i\sqrt{\lambda}a} & e^{-i\sqrt{\lambda}a} \\
e^{i\sqrt{\lambda}b} & e^{-i\sqrt{\lambda}b}
\end{vmatrix} \neq 0 \iff e^{i\sqrt{\lambda}(a-b)} \neq e^{-i\sqrt{\lambda}(a-b)} \iff e^{2i\sqrt{\lambda}(a-b)} \neq 1 \iff i\sqrt{\lambda}(a - b) \neq in\pi \iff \lambda \neq \frac{n^2 \pi^2}{(b - a)^2}, \quad n = 1, 2, \ldots
\]
If \(\lambda = 0\), we have \(y'' = 0\), so \(y = C_1 t + C_2 \Rightarrow\) only trivial solution.

In order to solve a BVP, it is often practical to use the following version of the method in the proof. Solve the two BVPs
\begin{align*}
(3) & \quad Ly = f, \quad By = 0, \\
(4) & \quad Ly = 0, \quad By = c.
\end{align*}
The sum of the solutions solves (1).
Green’s function

(4) can be solved by considering linear combinations of basis solutions. We now consider (3). The goal is to write the solution in the form

$$y(x) = \int_{a}^{b} G(x, \xi) f(\xi) \, d\xi$$

for some function $G(x, \xi)$.

We suppose for simplicity that the BCs are separated, i.e.

$$B_1 y = b_1^a y(a) + b_{12}^a y'(a) = 0,$$

$$B_2 y = b_2^b y(b) + b_{22}^b y'(b) = 0.$$

We suppress $a$ and $b$ in the coefficients $b_{jk}$.

Suppose that the equivalent conditions of the theorem are satisfied. Let $y_1$ be the solution of $L y = 0$ satisfying $y_1(a) = -b_{12}$, $y_1'(a) = b_{11}$ and $y_2$ the solution satisfying $y_2(b) = -b_{22}$, $y_2'(b) = b_{21}$. Then

$$B_1 y_1 = -b_1 b_{12} + b_{12} b_{11} = 0, \quad B_2 y_2 = -b_{21} b_{22} + b_{22} b_{21} = 0.$$

If $y_1 = c y_2$ for some $c \in \mathbb{C}$, then $B_2 y_1 = c B_2 y_2 = 0 \Rightarrow y_1$ nontrivial solution of the homogeneous problem. Contradiction! Hence, $\{y_1, y_2\}$ lin. indep.

Variation of parameters formula (ODE 1):

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \int_{a}^{x} \left[ y_2(x) y_1(\xi) - y_1(x) y_2(\xi) \right] f(\xi) a(\xi) W(\xi) \, d\xi,$$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = y_1(x) y'_2(x) - y'_1(x) y_2(x)$$

Wronskian of $y_1$ and $y_2$. $\dot{y}$ solution of ODE with $\dot{y}(a) = \dot{y}'(a) = 0$.

Apply $B_1$:

$$0 = B_1 y = C_1 B_1 y_1 + C_2 B_1 y_2 + B_1 \dot{y} = C_2 B_1 y_2.$$

But this implies $C_2 = 0$, since $B_1 y_2 = 0 \Rightarrow y_2$ nontrivial solution of the homogeneous problem. Hence,

$$y(x) = C_1 y_1(x) + \int_{a}^{x} y_2(x) y_1(\xi) - y_1(x) y_2(\xi) f(\xi) a(\xi) W(\xi) \, d\xi$$

$$= C_1 y_1(x) + \int_{a}^{x} y_2(x) y_1(\xi) a(\xi) W(\xi) \, d\xi + \int_{x}^{b} y_1(x) y_2(\xi) a(\xi) W(\xi) \, d\xi - \int_{a}^{b} y_1(x) y_2(\xi) a(\xi) W(\xi) \, d\xi$$

$$= \tilde{C}_1 y_1(x) + \int_{a}^{x} y_2(x) y_1(\xi) a(\xi) W(\xi) \, d\xi + \int_{x}^{b} y_1(x) y_2(\xi) a(\xi) W(\xi) \, d\xi,$$
where

\[ \tilde{C}_1 = C_1 - \int_a^b y_2(\xi) \frac{f(\xi)}{a_2(\xi)W(\xi)} \, d\xi. \]

We have

\[
y'(x) = \tilde{C}_1 y'_1(x) + y_2(x)y_1(x) \frac{f(x)}{a_2(x)W(x)} - y_1(x)y_2(x) \frac{f(x)}{a_2(x)W(x)} = 0 + \int_a^x y'_2(x)y_1(\xi) \frac{f(\xi)}{a_2(\xi)W(\xi)} \, d\xi + \int_x^b y'_1(x)y_2(\xi) \frac{f(\xi)}{a_2(\xi)W(\xi)} \, d\xi.
\]

Thus,

\[
y(b) = \tilde{C}_1 y_1(b) + y_2(b) \int_a^b y_1(\xi) \frac{f(\xi)}{a_2(\xi)W(\xi)} \, d\xi,
\]

\[
y'(b) = \tilde{C}_1 y'_1(b) + y'_2(b) \int_a^b y_1(\xi) \frac{f(\xi)}{a_2(\xi)W(\xi)} \, d\xi.
\]

It follows that

\[ 0 = B_2 y = \tilde{C}_1 B_2 y_1 \]

(since \( b_1 y_2(b) + b_2 y'_2(b) = 0 \)) and thus \( \tilde{C}_1 = 0 \). We have proved the following.

**Theorem.** If the BVP \( Ly = f, By = 0 \) has a unique solution, it is given by

\[ y(x) = \int_a^b G(x, \xi)f(\xi) \, d\xi, \]

where

\[ G(x, \xi) = \begin{cases} \frac{y_2(x)y_1(\xi)}{a_2(\xi)W(\xi)}, & a \leq \xi \leq x \leq b, \\ \frac{y_1(x)y_2(\xi)}{a_2(\xi)W(\xi)}, & a \leq x \leq \xi \leq b. \end{cases} \]

\( G \) is called Green’s function for the BVP (1).

**Example.** Consider the BVP \( y'' + y = f, y'(0) = y'(\frac{\pi}{2}) = 0 \). The general solution of the homogenous eq. is \( y(x) = \alpha_1 \cos x + \alpha_2 \sin x \). We have \( b_{11} = b_{21} = 0, b_{12} = b_{22} = 1 \), so \( y_1(0) = -1, y'_1(0) = 0 \), while \( y_2(\frac{\pi}{2}) = -1, y'_2(\frac{\pi}{2}) = 0 \). This gives

\[ y_1(x) = -\cos x \quad \text{and} \quad y_2(x) = -\sin x. \]

We have \( a_2(x) \equiv 1 \) and

\[ W(x) = \begin{vmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{vmatrix} = 1 \]

and thus

\[ G(x, \xi) = \begin{cases} \sin x \cos \xi, & 0 \leq \xi \leq \frac{\pi}{2}, \\ \sin \xi \cos x, & 0 \leq x \leq \xi \leq \frac{\pi}{2}. \end{cases} \]