Ordinary Differential Equations II

February 20

2018

Sturm-Liouville problems

Last time: the problem

\[-y''(x) = \lambda y(x), \quad y(0) = y(1) = 0,\]

has a nontrivial solution \( y(x) = c \sin(n\pi x), \ c \neq 0, \) iff \( \lambda = \lambda_n := n^2 \pi^2, \ n = 1, 2, 3, \ldots \) Any \( y \in C^2([0, 1], \mathbb{C}) \) with \( y(0) = y(1) = 0 \) can be written

\[
y(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x),
\]

for some \( c_n \in \mathbb{C}, \) where the (Fourier) series is uniformly convergent on \([0, 1]\).

The numbers \( \lambda_n \) can be seen as eigenvalues and the functions \( y_n(x) = \sin(n\pi x) \) as eigenvectors, or eigenfunctions, for the linear operator \( A := -\frac{d^2}{dx^2}. \)

Recall that a linear map \( A \) on an \( n \)-dim. inner product space \( V \) is symmetric (or Hermitian) if \( \langle Au, v \rangle = \langle u, Av \rangle \ \forall u, v \in V. \) If \( A \) is symmetric, then \( \exists \) ON basis of eigenvectors \( \{u_j\}_{j=1}^{n} \) (spectral theorem). Every vector \( u \in V \) can be written

\[
u = \sum_{j=1}^{n} c_j u_j
\]

where

\[
\langle u_j, u_k \rangle = \begin{cases} 1, & j = k, \\ 0, & j \neq k \end{cases} \quad \text{and} \quad c_j = \langle u_j, u \rangle.
\]

\( A = -\frac{d^2}{dx^2} \) is formally a symmetric operator with respect to the inner product

\[
\langle f, g \rangle = \int_{0}^{1} f^*(x)g(x) \, dx = \int_{0}^{1} \overline{f(x)}g(x) \, dx,
\]

\((** = \text{complex conjugation}), \) since

\[
\langle Af, g \rangle = \int_{0}^{1} -f''(x)g(x) \, dx = -[f'(x)g(x)]_0^1 + \int_{0}^{1} \overline{f'(x)}g'(x) \, dx = \left[ f(x)g'(x) \right]_0^1 - \int_{0}^{1} \overline{f(x)}g''(x) \, dx = \int_{0}^{1} f(x)(-g''(x)) \, dx = \langle f, Ag \rangle,
\]
provided that $f$ and $g$ vanish at the endpoints. Moreover, the eigenfunctions are orthogonal since

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) \, dx = \int_0^1 \frac{1}{2} (\cos((n-m)\pi x) - \cos((n+m)\pi x)) \, dx$$

$$= \frac{\sin((n-m)\pi)}{2(n-m)\pi} - \frac{\sin((n+m)\pi)}{2(n+m)\pi}$$

$$= 0$$

if $n \neq m$. The norm of $\sin(n\pi x)$ is $1/\sqrt{2}$. Setting $u_n(x) = \sqrt{2} \sin(n\pi x)$, $n = 1, 2, \ldots$ we thus get an ‘ON basis’ of eigenfunctions.

Goal: generalize this to eigenvalue problems of the form

$$-(p(x) y'(x))' + q(x) y(x) = \lambda r(x) y(x), \quad a \leq x \leq b,$$

with boundary conditions $b_{11} y(a) + b_{12} y'(a) = b_{21} y(b) + b_{22} y'(b) = 0$. This is called a Sturm-Liouville problem. Can be written $Ay = \lambda y$ with

$$A = \frac{1}{r(x)} \left( -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right).$$

**Example.** Consider a string with varying density. The equation for small vibrations is then given by the variable wave eq.

$$\frac{\partial^2 u}{\partial t^2}(t, x) = c^2(x) \frac{\partial^2 u}{\partial x^2}(t, x).$$

Separation of variables leads to the eigenvalue problem

$$-y'' = \lambda e^{-2}(x) y,$$

so that $p(x) \equiv 1$, $q(x) \equiv 0$ and $r(x) = e^{-2}(x)$.

We first need to discuss some general properties of inner product spaces.

**Inner product spaces**

$H$ complex vector space. A map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ is called an inner product if

1. $\langle f, f \rangle > 0$, $f \neq 0$ (positive definiteness),

2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$ (Hermitian symmetry),

3. $\langle f, \alpha_1 g_1 + \alpha_2 g_2 \rangle = \alpha_1 \langle f, g_1 \rangle + \alpha_2 \langle f, g_2 \rangle$ (linearity in 2nd argument).

Remarks:

1. $\langle 0, 0 \rangle = 0$. 

2.
(2) and (3) \[ \langle \alpha_1 f_1 + \alpha_2 f_2, g \rangle = \overline{\alpha_1} \langle f_1, g \rangle + \overline{\alpha_2} \langle f_2, g \rangle. \]

An inner product is linear in the 2nd argument and conjugate linear in the 1st. This is called sesquilinearity (generalization of bilinearity). Usual convention in physics; in mathematics usually the opposite.

We define the norm corresponding to \( \langle \cdot, \cdot \rangle \) by
\[
\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}.
\]

Examples.
- \( \mathbb{C}^n \) with \( \langle a, b \rangle = \sum_{j=1}^{n} \overline{a_j} b_j \) and \( \| a \| = \sqrt{\sum_{j=1}^{n} |a_j|^2} \).
- \( l^2 = \{ u = \{ u_j \}_{j=1}^{\infty} : \sum_{j=1}^{\infty} |u_j|^2 < \infty \} \) with \( \langle u, v \rangle = \sum_{j=1}^{\infty} \overline{u_j} v_j \) and \( \| u \| = \sqrt{\sum_{j=1}^{\infty} |u_j|^2} \).
- \( C([a, b], \mathbb{C}) \) with \( \langle f, g \rangle = \int_{a}^{b} \overline{f(x)} g(x) \, dx \) and \( \| f \| = \sqrt{\int_{a}^{b} |f(x)|^2 \, dx} \).

If \( H \) is complete with the norm coming from the inner product (a Banach space), it’s called a Hilbert space. We won’t require this.

We write \( f \perp g \) if \( \langle f, g \rangle = 0 \). Recall that
\[
\| f + g \|^2 = \| f \|^2 + \| g \|^2, \quad f \perp g.
\]

We also have the Cauchy-Schwarz inequality (Thm 5.2)
\[
|\langle f, g \rangle| \leq \| f \| \| g \|.
\]

This yields the triangle inequality for \( \| \cdot \| \), showing that it’s indeed a norm:
\[
\| f + g \|^2 = \langle f + g, f + g \rangle = \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle
\]
\[
\leq \| f \|^2 + 2\| f \| \| g \| + \| g \|^2 = (\| f \| + \| g \|)^2.
\]

A set of vectors \( \{ u_j \} \) is called an orthonormal set if
\[
\langle u_j, u_k \rangle = \begin{cases} 
1, & j = k, \\
0, & j \neq k.
\end{cases}
\]

We can define the orthogonal projection on an ON set using the following lemma.

Lemma. Suppose \( \{ u_j \}_{j=0}^{n} \) is an ON set and let \( V \) be the span of \( \{ u_j \}_{j=0}^{n} \). Every \( f \in H \) can be written
\[
f = f_\parallel + f_\perp = \sum_{j=0}^{n} \langle u_j, f \rangle u_j,
\]
with \( f_\parallel \perp f_\perp \). Moreover, \( \langle u_j, f_\perp \rangle = 0 \) for all \( j \) and
\[
(2) \quad \| f \|^2 = \sum_{j=0}^{n} |\langle u_j, f \rangle|^2 + \| f_\perp \|^2.
\]

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Every $\hat{f}$ in $V$ satisfies
\[ \|f - \hat{f}\| \geq \|f_\perp\| \]
with equality iff $\hat{f} = f_\parallel$. That is, $f_\parallel$ is the unique vector in $V$ closest to $f$.

Proof.
\[
\langle u_j, f_\perp \rangle = \langle u_j, f - f_\parallel \rangle = \langle u_j, f \rangle - \langle u_j, f_\parallel \rangle = \langle u_j, f \rangle - \left( u_j, \sum_{k=0}^{n} \langle u_k, f \rangle u_k \right) \\
= \langle u_j, f \rangle - \sum_{k=0}^{n} \langle u_k, f \rangle \langle u_j, u_k \rangle = \langle u_j, f \rangle - \langle u_j, f \rangle = 0.
\]
In particular, $\langle f_\parallel, f_\perp \rangle = 0$ since $f_\parallel$ is in $V$. Thus,
\[ \|f\|^2 = \|f_\parallel\|^2 + \|f_\perp\|^2 \]
with
\[ \|f_\parallel\|^2 = \left( \sum_{j=0}^{n} \langle u_j, f \rangle u_j, \sum_{k=0}^{n} \langle u_k, f \rangle u_k \right) = \sum_{j=0}^{n} \sum_{k=0}^{n} \langle u_j, f \rangle \langle u_k, f \rangle \langle u_j, u_k \rangle = \sum_{j=0}^{n} |\langle u_j, f \rangle|^2. \]
Fix a vector $\hat{f} = \sum_{j=0}^{n} \alpha_j u_j$ in $V$. Then
\[ \|f - \hat{f}\|^2 = \|f_\parallel + f_\perp - \hat{f}\|^2 = \|f_\perp\|^2 + \|f_\parallel - \hat{f}\|^2, \]
so that
\[ \|f - \hat{f}\| \geq \|f_\perp\| \]
with equality iff $\hat{f} = f_\parallel$. \qed

(2) $\Rightarrow$ Bessel’s inequality:
\[ \sum_{j=0}^{n} |\langle u_j, f \rangle|^2 \leq \|f\|^2 \]
with equality iff $f$ is in the span of $\{u_j\}_{j=0}^{n}$.

We assume that $H$ is infinite-dimensional. An orthonormal set $\{u_j\}_{j=0}^{\infty}$, is called an orthonormal basis if
\[ \|f\|^2 = \sum_{j=0}^{\infty} |\langle u_j, f \rangle|^2 \ \forall f \in H. \]
Set
\[ f_n = \sum_{j=0}^{n} \langle u_j, f \rangle u_j. \]
If $\{u_j\}$ is an ON set then (2) $\Rightarrow$
\[ \|f - f_n\|^2 = \|f\|^2 - \sum_{j=0}^{n} |\langle u_j, f \rangle|^2. \]
Hence, \( f_n \to f \) in \( H \), i.e.
\[
f = \sum_{j=0}^{\infty} \langle u_j, f \rangle u_j
\]
iff \( \{u_j\} \) is an ON basis.

A linear operator on \( H \) is simply a linear map \( A : H \to H \). It’s useful to extend this definition so that \( A \) only is defined on some subspace \( D(A) \) of \( H \) (with values in \( H \)). The space \( D(A) \) is called the domain of \( A \).

A is called symmetric if
\[
\langle Af, g \rangle = \langle f, Ag \rangle, \quad \forall f, g \in D(A)
\]
(the book also requires that \( D(A) \) is dense in \( H \)).

**Example.** The operator \( A = -\frac{d^2}{dx^2} \) can be be considered as a linear operator on \( H = C([a, b], \mathbb{C}) \) with \( D(A) = \{ f \in C^2([a, b], \mathbb{C}) : f(a) = f(b) = 0 \} \). Our previous calculation shows that \( A \) is symmetric.

\( \lambda \in \mathbb{C} \) is called an **eigenvalue** if \( \exists \) nonzero eigenvector \( u \in D(A) \), s.t. \( Au = \lambda u \).

**Theorem.** Let \( A \) be symmetric. Then all eigenvalues are real and eigenvectors corresponding to different eigenvalues are orthogonal.

**Proof.** Suppose that \( u \) is an eigenvector corresponding to \( \lambda \). Then
\[
\lambda \langle u, u \rangle = \langle u, Au \rangle = \langle Au, u \rangle = \overline{\lambda} \langle u, u \rangle
\]
\( \Rightarrow \lambda = \overline{\lambda} \) since \( \langle u, u \rangle = \|u\|^2 \neq 0 \). That is, \( \lambda \) is real.

If \( Au_j = \lambda_j u_j, \ j = 1, 2, \lambda_1 \neq \lambda_2 \), then
\[
\lambda_2 \langle u_1, u_2 \rangle = \langle u_1, Au_2 \rangle = \langle Au_1, u_2 \rangle = \lambda_1 \langle u_1, u_2 \rangle
\]
since \( \lambda_1 \in \mathbb{R} \). Since \( \lambda_1 \neq \lambda_2 \) we obtain \( \langle u_1, u_2 \rangle = 0 \).

A linear operator \( A \) with \( D(A) = H \) is called **bounded** if
\[
\|A\| := \sup_{\|f\|=1} \|Af\| < \infty.
\]
This is called the operator norm of \( A \). Defines a norm on the vector space of bounded operators (exercise). If \( f \neq 0 \), then \( \|f\|^{-1} f \) is a unit vector, so
\[
\|A(\|f\|^{-1} f)\| \leq \|A\| \Rightarrow \|Af\| \leq \|A\|\|f\|.
\]
The last inequality holds trivially if \( f = 0 \). This gives
\[
\|Af - Ag\| = \|A(f - g)\| \leq \|A\|\|f - g\|,
\]
so \( A \) is (Lipschitz) continuous.

An operator \( A \) with \( D(A) = H \) is called compact if every sequence \( \{Af_n\} \) has a convergent subsequence whenever \( \{f_n\} \) is a bounded sequence. Every compact operator is bounded (exercise).

Next time: spectral thm for compact symmetric operators.