Ordinary Differential Equations II

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Sturm-Liouville problems

(Regular) Sturm-Liouville problem

\[-(p(x)y'(x))' + q(x)y(x) = \lambda r(x)y(x), \quad a < x < b,\]

\[b_{11}y(a) + b_{12}y'(a) = 0,\]

\[b_{21}y(b) + b_{22}y'(b) = 0.\]

\[p \in C^1([a, b], \mathbb{R}), \quad q, r \in C([a, b], \mathbb{R}), \quad p(x), r(x) > 0. \]

We assume \((b_{11}, b_{12}) \neq (0, 0)\) and \((b_{21}, b_{22}) \neq (0, 0)\) and real. Can rewrite the BCs as

\[BC_a(y) := \cos(\alpha) y(a) - \sin(\alpha) p(a) y'(a) = 0,\]

\[BC_b(y) := \cos(\beta) y(b) - \sin(\beta) p(b) y'(b) = 0,\]

for some \(\alpha, \beta \in \mathbb{R}\), where

\[
\cos(\alpha) = \frac{b_{11}}{\sqrt{b_{11}^2 + p(a)^2 b_{12}^2}}, \quad \sin(\alpha) = -p(a)^{-1}b_{12}/\sqrt{b_{11}^2 + p(a)^2 b_{12}^2},
\]

etc.

Eigenvalue problem for the operator

\[L := \frac{1}{r(x)} \left( -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right)\]

on \(H = C([a, b], \mathbb{C})\) with inner product \(\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) r(x) \, dx\). \(D(L) := \{f \in C^2([a, b], \mathbb{C}) : BC_a(f) = BC_b(f) = 0\}\).

**Proposition.** \(D(L)\) is dense in \(H\).

*Proof.* See Techl, Lemma 5.8, p. 156.

**Proposition.** \(L\) is symmetric.
Proof.

\[ \int_a^b g(x)(Lf)(x)r(x) \, dx = \int_a^b g(x)\left(-(p(x)f'(x))' + q(x)f(x)\right) \, dx \]
\[ = \int_a^b \left(p(x)g'(x)f'(x) + q(x)g(x)f(x)\right) \, dx + \left[-p(x)g(x)f'(x)\right]_a^b \]
\[ = \int_a^b \left(-(p(x)g'(x))' + q(x)g(x)f(x)\right) \, dx \]
\[ + \left[p(x)(g'(x)f(x) - g(x)f'(x))\right]_a^b \]
\[ = \int_a^b (Lg)(x)f(x)r(x) \, dx + W_a(g, f) - \tilde{W}_b(g, f), \]

where

\[ \tilde{W}_x(g, f) = p(x)(g(x)f'(x) - g'(x)f(x)) = p(x) \left| \begin{array}{c} g(x) \\ g'(x) \\ f(x) \
\end{array} \right| \]

is the modified Wronskian for the Sturm-Liouville problem. If \( f, g \in D(L) \), then \((f(a), f'(a))\) and \((g(a), g'(a))\) are lin. dep. The same is true at \( x = b \). Hence, \( W_a(g, f) = \tilde{W}_b(g, f) = 0 \).

\[ \square \]

Goal for today:

**Theorem 1.** The Sturm-Liouville problem has a sequence of real simple eigenvalues \( \{\lambda_n\}_{n=0}^\infty \), with \( \lim_{n \to \infty} |\lambda_n| = \infty \). The corresponding normalized eigenfunctions \( u_n \) can be chosen real-valued and form an ON basis for \( H \). That is, every \( f \in H \) can be written

\[ f(x) = \sum_{n=0}^\infty \langle u_n, f \rangle u_n(x) \]

with convergence in \( H \). For \( f \in D(L) \), the series converges uniformly on \([a, b] \).

Idea: use the spectral theorem for compact symmetric operators. \( L \) is not compact, but its inverse is if it exists. Of course, \( L^{-1} \) doesn’t exist if 0 is an eigenvalue of \( L \). Instead, use the resolvent \( R_L(\lambda) := (L - \lambda I)^{-1} \), \( \lambda \) not an eigenvalue of \( L \).

Note that the problem

\( (L - \lambda I)y = f, \)

\( y \in D(L), \ f \in H, \) can be written

\[ (1) \quad a_2(x)y'' + a_1(x)y' + a_0(x)y = f, \quad BC_a(y) = BC_a(y) = 0, \]

with

\[ a_2(x) = -\frac{p(x)}{r(x)}, \quad a_1(x) = -\frac{p'(x)}{r(x)}, \quad a_0(x) = \frac{q(x) - \lambda r(x)}{r(x)}. \]

(1) uniquely solvable for all \( f \iff (1) \) with \( f = 0 \) only has the trivial solution \( \iff \lambda \) is not an eigenvalue. If \( \lambda \) is not an eigenvalue, then

\[ y(x) = \int_a^b G(\lambda, x, \xi)f(\xi) \, d\xi, \]
where $G(\lambda, x, \xi)$ is Green’s function (depending on the parameter $\lambda \in \mathbb{C}$). Recall that

$$
G(\lambda, x, \xi) = \begin{cases}
u_u(\lambda, \xi)u_u(\lambda, x), & \xi \leq x, \\
\nu_u(\lambda, x)u_u(\lambda, \xi), & x \leq \xi,
\end{cases}
$$

where $u_a$ and $u_b$ are non-trivial solutions of the homogeneous equation satisfying the BCs on the left and right, respectively, while $W(\lambda, \xi)$ is the Wronskian of $u_a$ and $u_b$. Concretely:

$$
u_a(\lambda, a) = \sin \alpha, \quad p(a)\nu'_a(\lambda, a) = \cos \alpha,
$$

$$
u_b(\lambda, b) = \sin \beta, \quad p(b)\nu'_b(\lambda, b) = \cos \beta.
$$

Teschl:

$$
R_L(\lambda)f := \int_a^b G(\lambda, x, \xi)f(\xi)r(\xi) \, d\xi;
$$

$$
\tilde{G}(\lambda, x, \xi) = \frac{G(\lambda, x, \xi)}{r(\xi)} = \begin{cases}
u_u(\lambda, \xi)u_u(\lambda, x), & \xi \leq x, \\
\nu_u(\lambda, x)u_u(\lambda, \xi), & x \leq \xi,
\end{cases}
$$

and $\tilde{W} = -pW$ indep. of $\xi$.

If we look at $L - \lambda I$ as an operator $D(L) \to H$, then the operator

$$
R_L(\lambda)f := \int_a^b G(\lambda, x, \xi)f(\xi) \, d\xi;
$$

is the inverse of $L - \lambda I$. $R_L(\lambda): H \to D(L)$ is called the resolvent of $L$.

Since $G(\lambda, \cdot, \cdot)$ is continuous on $[a, b] \times [a, b]$, $R_L$ is a compact operator on $H$.

If $\lambda \in \mathbb{R}$ we have

$$
\langle R_Lf, g \rangle = \langle R_Lf, (L - \lambda I)R_Lg \rangle = \langle (L - \lambda I)R_Lf, R_Lg \rangle = \langle f, R_Lg \rangle,
$$

since $L$ is symmetric. Can also show this directly using Green’s function (Teschl).

We already know that all eigenvalues of $L$ are real. Hence, $R_L(\lambda)$ exists and is compact for $\lambda \not\in \mathbb{R}$, but unfortunately it’s not symmetric then.

Teschl uses complex analysis to show that there $\lambda \in \mathbb{R}$ which are not eigenvalues (Lemma 5.7 and 5.9). Here’s a proof which doesn’t require complex analysis.

**Lemma.** The eigenvalues of $L$ are at most countably many and have no finite accumulation point.
Proof. We already know that the eigenvalues are real. Suppose that \( \exists \) sequence of (distinct) eigenvalues \( \{\lambda_n\} \) with \( \lim_{n \to \infty} \lambda_n = \lambda \in \mathbb{R} \) and let \( \{u_n(x)\} \) be the corresponding eigenfunctions (can also be assumed to be real). Let \( y_1(\lambda, x), y_2(\lambda, x) \) be linearly independent solutions of the homogeneous eq. \( Ly = \lambda y \), \( y \) with fixed initial conditions at \( y = a \), e.g. \( y_1(\lambda, a) = 1, y_1'(\lambda, a) = 0, y_2(\lambda, a) = 0, y_2'(\lambda, a) = 1 \). Then
\[
   u_n(x) = c_{1,n} y_1(\lambda_n, x) + c_{2,n} y_2(\lambda_n, x),
\]
with \((c_{1,n}, c_{2,n}) \neq (0, 0)\). After multiplying \( u_n \) with a constant, we can assume that \( c_{1,n}^2 + c_{2,n}^2 = 1 \). By Bolzano-Weierstrass we can find subsequences with \( c_{1,n_k} \to c_1 \), \( c_{1,n_k} \to c_2 \), where \( c_1^2 + c_2^2 = 1 \). By continuous dependence on parameters, we also have that \( y_j(\lambda_{n_k}, x) \to y_j(\lambda, x) \), \( j = 1, 2 \), uniformly on \([a, b]\). Hence, \( u_{n_k} \to u \) uniformly, where
\[
   u(x) = c_1 y_1(\lambda, x) + c_2 y_2(\lambda, x) \neq 0.
\]
On the other hand,
\[
   \langle u_{n_k}, u_{n_{k+1}} \rangle = 0
\]
since eigenfunctions corresponding to different eigenvalues are orthogonal. Letting \( k \to \infty \) gives
\[
   \|u\|^2 = \langle u, u \rangle = 0,
\]
so that \( u(x) \equiv 0 \). Contradiction!

Proof of Thm 1. Choose a number \( \lambda \in \mathbb{R} \) which is not an eigenvalue of \( L \). Then \( R_L(\lambda) \) exists, and is a compact and symmetric operator on \( H \). Note that
\[
   Lu = zu \Leftrightarrow (L - \lambda)u = (z - \lambda)u \Leftrightarrow u = (z - \lambda)R_L(\lambda)u
\]
\[
   \Leftrightarrow R_L(\lambda)u = \frac{1}{z - \lambda}u, \quad z \neq \lambda.
\]
Hence \( z \neq \lambda \) is an eigenvalue of \( L \) iff \( 1/(z - \lambda) \) is an eigenvalue of \( R_L \), and they have the same eigenvectors.

By the spectral theorem for compact symmetric operators, \( R_L(\lambda) \) has a sequence of real non-zero eigenvalues \( \{\alpha_n\} \) with \( \lim_{n \to \infty} \alpha_n = 0 \). Hence, \( L \) has a sequence of real eigenvalues \( \{\lambda_n\}_{n=0}^{\infty} \), with \( \lambda_n = \lambda + \frac{1}{\alpha_n} \) and thus \( |\lambda_n| \to \infty \) as \( n \to \infty \). The corresponding eigenvectors \( \{u_n\}_{n=0}^{\infty} \) can be assumed to be real since the coefficients of \( L \) are real.

Since \( \text{Ran}(R_L(\lambda)) = D(L) \) is dense in \( H \), the eigenfunctions form an ON basis for \( H \).

The eigenvalues are simple, since if \( u_n \) and \( v_n \) are two different eigenfunctions corresponding to \( \lambda_n \), then \( BC_a(u_n) = BC_a(v_n) = 0 \Rightarrow u_n, v_n \) lin. dep.

Suppose that \( f \in D(L) \). We need to show that the eigenfunction expansion converges uniformly to \( f \) on \([a, b]\). It suffices to show that it converges uniformly, since if it converges uniformly to \( \tilde{f} \), then it also converges to \( \tilde{f} \) in \( H \), as follows by the inequality
\[
   \int_a^b |u(x)|^2 r(x) \, dx \leq C \sup_{a \leq x \leq b} |u(x)|^2,
\]

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with \( C = (b - a)^{1/2} \sup_{a \leq x \leq b} r(x) \). Write \( D(L) \ni f = R_L(\lambda)g, \ g \in H \). Then
\[
\sum_{j=0}^{\infty} \langle u_j, f \rangle u_j(x) = \sum_{j=0}^{\infty} \langle u_j, R_L(\lambda)g \rangle u_j(x) = \sum_{j=0}^{\infty} \langle R_L(\lambda)u_j, g \rangle u_j(x) \\
= \sum_{j=0}^{\infty} \alpha_j \langle u_j, g \rangle u_j(x).
\]

Cauchy-Schwarz \( \Rightarrow \)
\[
\sum_{j=m}^{n} \alpha_j \langle u_j, g \rangle u_j(x) \leq \sum_{j=m}^{n} |\langle u_j, g \rangle|^2 \sum_{j=m}^{n} |\alpha_j u_j(x)|^2.
\]

\( \{u_j\} \) ON basis \( \Rightarrow \sum_{j=0}^{\infty} |\langle u_j, g \rangle|^2 = \|g\|^2 < \infty. \)
\[
\alpha_j u_j(x) = R_L(\lambda)u_j(x) = \int_a^b \hat{G}(\lambda, x, \xi)u_j(\xi)r(\xi) \, d\xi = \langle u_j, \hat{G}(\lambda, x, \cdot) \rangle.
\]

Hence,
\[
\sum_{j=m}^{n} |\alpha_j u_j(x)|^2 \leq \sum_{j=0}^{\infty} |\langle u_j, \hat{G}(\lambda, x, \cdot) \rangle|^2 = \int_a^b |\hat{G}(\lambda, x, \xi)|^2 r(\xi) \, d\xi < \infty
\]
(independent of \( x \)). It follows that the RHS of (2) \( \to 0 \) uniformly as \( m, n \to \infty \). Hence, the LHS is a Cauchy sequence w.r.t. the sup norm. Since \( C([a, b], \mathbb{C}) \) is complete, the series \( \sum_{j=0}^{\infty} \alpha_j \langle u_j, g \rangle u_j(x) = \sum_{j=0}^{\infty} \langle u_j, f \rangle u_j(x) \) converges uniformly. \( \Box \)

**Example.**
\[-y'' = \lambda y, \quad y(0) = y(1) = 0, \]
Eigenvalues \( \lambda_n = n^2 \pi^2, \ n = 1, 2, 3, \ldots \) Normalized eigenfunctions
\[
u_n(x) = \sqrt{2} \sin(n\pi x).
\]

Any \( f \in C([0, 1], \mathbb{C}) \) can be written
\[
f(x) = \sum_{n=1}^{\infty} c_n u_n(x) = \sqrt{2} \sum_{n=1}^{\infty} c_n \sin(n\pi x)
\]
with convergence w.r.t. the norm \( \sqrt{\int_0^1 \|f(x)\|^2 \, dx} \), where
\[
c_n = \langle u_n, f \rangle = \int_0^1 u_n(x) f(x) \, dx = \sqrt{2} \int_0^1 \sin(n\pi x) f(x) \, dx.
\]

We also have that
\[
\int_0^1 \|f(x)\|^2 \, dx = \sum_{n=1}^{\infty} |c_n|^2.
\]
If $f \in C^2$ and $f(0) = f(1) = 0$, then (3) converges uniformly.  

Take $f(x) \equiv 1$. Then

$$c_n = \sqrt{2} \int_0^1 \sin(n\pi x) \, dx = \frac{\sqrt{2}}{n\pi} [-\cos(n\pi x)]_0^\pi = \frac{2\sqrt{2}}{n\pi}$$

if $n$ is odd, and 0 if $n$ is even. Hence,

$$1 = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin((2k+1)\pi x),$$

with convergence in mean square. The convergence is not uniform on $[0, 1]$ (why?).  

We also have that

$$1 = \int_0^1 x^2 \, dx = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2 \pi^2}$$

or

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$