Linear autonomous systems

\[ x' = Ax, \quad x(0) = x_0 \]

A \( n \times n \) matrix.

Solution:

\[ x(t) = e^{tA}x_0 = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} x_0. \]

Recall that

\[ e^0 = I, \]
\[ (e^A)^{-1} = e^{-A}, \]
\[ e^A e^B = e^{A+B}, \quad \text{if } AB = BA. \]

If \( v \) is an eigenvector corresponding to the eigenvalue \( \lambda \), then

\[ e^{tA}v = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} v = \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} v = e^{t\lambda}v. \]

If \( v \) is a generalized eigenvector, say \( (A - \lambda I)^m v = 0 \), then

\[ e^{tA}v = e^{t\lambda} e^{t(A - \lambda I)}v = e^{t\lambda} \sum_{k=0}^{m-1} \frac{(A - \lambda I)^k t^k}{k!} v. \]

Recall that there always exists a basis consisting of generalized eigenvectors. We can calculate \( e^{tA}x_0 \) by expressing \( x_0 \) in this basis and using linearity.

More generally, we can compute \( e^{tA} \) by expressing each element of the standard basis \( \{e_1, \ldots, e_n\} \) in a basis of generalized eigenvectors.

Matrix interpretation:

Suppose that \( v_1, \ldots, v_n \) is a basis of generalized eigenvectors and let \( T \) be the \( n \times n \) matrix whose \( j \)th column contains the coordinates for \( v_j \) in the standard
basis. Then $B = T^{-1}AT$ is the matrix for the linear map $x \mapsto Ax$ in the new basis and

$$e^{tA} = Te^{tB}T^{-1}.$$ 

Suppose that $A$ is diagonalizable, i.e. there’s a basis consisting of eigenvectors. Then

$$B = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

and

$$e^{tB} = \text{diag}(e^{t\lambda_1}, \ldots, e^{t\lambda_n}).$$

In general, the matrix $B$ will only be block diagonal,

$$B = \begin{pmatrix} B_1 & & \\ & B_2 & \\ & & \ddots \\ & & & B_l \end{pmatrix}$$

and

$$e^{tB} = \begin{pmatrix} e^{tB_1} & & \\ & e^{tB_2} & \\ & & \ddots \\ & & & e^{tB_l} \end{pmatrix}$$

where each $B_j$ can be written $\lambda_j I + N_j$ for some nilpotent matrix $N_j$. This gives

$$e^{tB_j} = e^{t\lambda_j t} \sum_{k=0}^{m_j-1} \frac{t^k N_j^k}{k!}.$$ 

In particular, one can find a basis such that each block has the form

$$\begin{pmatrix} \lambda_j & 1 & 0 & \ldots & 0 \\ 0 & \lambda_j & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_j \end{pmatrix}.$$ 

This is called the Jordan normal form (or Jordan canonical form) of $A$. Section 3.8 contains more on this subject, but is not required reading.

Main point:
The entries of $e^{tA}$ are linear combinations of terms $t^k e^{t\lambda_j}$. If the algebraic and geometric multiplicities of $\lambda_j$ are equal, $k = 0$.

**Linear autonomous systems in the plane**

$$x' = Ax$$

$A \in \mathbb{R}^{2 \times 2}$, $x \in \mathbb{R}^2$, det $A \neq 0$. $x = 0$ is a fixed point (constant solution).

$\lambda_1, \lambda_2$ eigenvalues of $A$ (we allow $\lambda_1 = \lambda_2$).
Case 1 (\(A\) diagonalisable, \(\lambda_1, \lambda_2 \in \mathbb{R}\))

\(u_1, u_2\) corresponding eigenvectors.
Write \(x_0 = c_1 u_1 + c_2 u_2\) and \(x(t) = y_1(t)u_1 + y_2(t)u_2\). Then
\[x(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2.\]
Thus, \(y_1(t) = c_1 e^{\lambda_1 t}, y_2(t) = c_2 e^{\lambda_2 t}\).
\[y_2 = c|y_1|^\gamma, \quad \gamma = \frac{\lambda_2}{\lambda_1}, c = \pm c_2/|c_1|^\gamma\]
or
\[y_1 = 0 \quad (if \ c_1 = 0).\]

Phase portraits:

Case 2 (\(A\) diagonalisable, \(\lambda_1, \lambda_2 \not\in \mathbb{R}\))

\[Au = \lambda u \Rightarrow A\bar{u} = \bar{A}u = \bar{\lambda}u.\]

Eigenvalues \(\lambda, \bar{\lambda}\), eigenvectors \(u, \bar{u}\).
Write
\[x(t) = c_1 e^{\lambda t} u + c_2 e^{\bar{\lambda} t} \bar{u}.\]
Write
\[\lambda = \sigma + i\omega, \quad u = v + i w.\]

\[x(t) = c_1 e^{\lambda t} u + c_2 e^{\bar{\lambda} t} \bar{u}\]
\[= c_1 e^{\sigma t + i\omega t} (v + i w) + c_2 e^{\bar{\sigma} t - i\omega t} (v - i w)\]
\[= e^{\sigma t}((c_1 e^{i\omega t} + c_2 e^{-i\omega t}) v + i(c_1 e^{i\omega t} - c_2 e^{-i\omega t}) w)\]
\[= e^{\sigma t}(((c_1 + c_2) \cos(\omega t) + i(c_1 - c_2) \sin(\omega t)) v\]
\[+ (i(c_1 - c_2) \cos(\omega t) - (c_1 + c_2) \sin(\omega t)) w)\]
\[= e^{\sigma t}((A \cos(\omega t) + B \sin(\omega t)) v + (B \cos(\omega t) - A \sin(\omega t)) w),\]
with $A, B \in \mathbb{R}$ (since $x(t) \in \mathbb{R}^2$). Can be written

$$x(t) = Ce^{\sigma t}(\cos(\omega t + \theta)v - \sin(\omega t + \theta)w),$$

with $C = \sqrt{A^2 + B^2}$.

$$y(t) = Ce^{\sigma t}(\cos(\omega t + \theta), -\sin(\omega t + \theta))$$

$y(t) = (y_1(t), y_2(t))$ coordinates in basis $v, w$.

\begin{center}
\begin{tikzpicture}
\begin{scope}[scale=0.5]
\draw (-2,0) -- (2,0) node[above] {$v_2$} -- (0,-2) node[right] {$v_1$};
\draw[->] (-2,0) arc (180:0:2);
\draw[->] (0,-2) arc (270:90:2);
\filldraw (0,0) circle (2pt);
\node at (-2,0) {Center};
\node at (0,-2) {Center};
\end{scope}
\end{tikzpicture}
\end{center}

**Case 3 ($A$ not diagonalisable)**

Let $\lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$.

Choose $u_1$ eigenvector, $u_2 \parallel u_1$. Then $u_2$ is a generalized eigenvector and $u_1, u_2$ a basis of generalized eigenvectors. Since $(A - \lambda I)^2 u_2 = 0$, we have $(A - \lambda I)u_2 = au_1$ for some $a \in \mathbb{R}$. Can suppose $a = 1$ (otherwise, divide $u_2$ by $a$). We have $e^{At}u_1 = e^{\lambda t}u_1$ and

$$e^{At}u_2 = e^{\lambda t}(I + t(A - \lambda I))u_2 = e^{\lambda t}(u_2 + tu_1).$$

If $x_0 = c_1u_1 + c_2u_2$ we have

$$e^{tA}x_0 = e^{\lambda t}((c_1 + c_2t)u_1 + c_2u_2).$$

If $\lambda < 0$ all solutions converge to $0$ as $t \to \infty$ (sink). If $\lambda > 0$ they grow exponentially as $t \to \infty$ (source). The factor $t$ doesn’t matter much. See Figure 3.5 in Teschl.

**Example.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = (1 - \lambda)^2 - 4 \Rightarrow \lambda_1 = -1, \lambda_2 = 3.$$ 

Eigenvectors $u_1 = (1, -1)$, $u_2 = (1, 1)$.

Saddle point.
Stability

Generally, \( x(t) \) is a linear combination of terms of the form \( t^k e^{\lambda t} \). No \( t^k \) if \( g_j = a_j \) (geom. mult. equals alg. mult.).

- \( \text{Re} \lambda_j < 0 \ \forall j \Rightarrow \) all solutions converge to 0 as \( t \to \infty \). We say that the system is asymptotically stable.
- If the solutions are bounded for \( t \geq 0 \), we say that the system is stable. This is the case if \( \text{Re} \lambda_j \leq 0 \ \forall j \) and \( g_j = a_j \) if \( \text{Re} \lambda_j = 0 \).
- If there exits an unbounded solution, the system is called unstable. This is the case if \( \text{Re} \lambda_j > 0 \) or for some \( j \), or \( \text{Re} \lambda_j = 0 \) and \( g_j < a_j \) for some \( j \).

Note that
\[
\left| \frac{t^k e^{\lambda t}}{e^{\alpha t}} \right| = t^k e^{(\text{Re} \lambda - \alpha) t} \to 0
\]
as \( t \to \infty \) if \( \alpha > \text{Re} \lambda \). This yields the following result.

**Proposition.** If the system is stable, \( \exists C \geq 0 \), s.t.
\[
\| e^{tA} \| \leq C, \quad t \geq 0.
\]

If the system is asymptotically stable, then for any \( \alpha > 0 \) with \( -\alpha > \text{Re}(\lambda_j) \) for all eigenvalues \( \lambda_j \) of \( A \), \( \exists C_\alpha \geq 0 \) s.t.
\[
\| e^{tA} \| \leq C_\alpha e^{-\alpha t}, \quad t \geq 0.
\]

**Inhomogeneous systems**

Recall that the solution of
\[
x' = Ax + g(t), \quad x(0) = x_0
\]
is given by Duhamel’s formula:
\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}g(s) \, ds.
\]

We will use this later when discussing stability for nonlinear systems.