Exercises - Set 1

1. Problem 2.6 on p. 39.

2. Investigate if \( f(t, x) \) is locally Lipschitz continuous in the second argument near the origin. If yes, find a Lipschitz constant for the set \([0, 1] \times [0, 1]\).
   - a) \( f(t, x) = t^2 + x^2 \)
   - b) \( f(t, x) = \sin t \cdot \cos x \)
   - c) \( f(t, x) = |t - x| \)
   - d) \( f(t, x) = |x|^\alpha \) \((\alpha > 0)\).

3. Problem 2.7 on p. 39.

4. Problem 2.8 on p. 39.

5. Problem 2.9 on p. 39.

6. Show that the solution \( x(t) \) of the IVP
   \[ x' = t^2 + e^{-x^2}, \quad x(0) = 0 \]
   is defined in the interval \( 0 \leq t \leq 1/2 \) and satisfies \( |x(t)| \leq 1 \) there.

7. Show that the solution of
   \[ x' = \sin t + \ln(1 + x^2), \quad x(0) = 0 \]
   is defined on the whole real axis.

8. We obtain an approximative solution of
   \[ x' = \sin(tx), \quad x(0) = 0.2 \]
   by instead solving
   \[ x' = tx, \quad x(0) = 0.2 \]
   Estimate the difference between the two solutions when \( t = 0.5 \).
   Hint: \( |\sin y - y| \leq \frac{|y|^3}{3!} \) for all \( y \).

9. Problem 2.10 on p. 42.

10. Give an alternative proof of the Picard-Lindelöf theorem without the condition \( T_0 < 1/L \) by replacing the norm \( \|x(t)\| = \sup_{0 \leq t \leq T_0} |x(t)| \) with the norm \( \|x(t)\|_\alpha = \sup_{0 \leq t \leq T_0} e^{-\alpha t} |x(t)| \), where \( \alpha > L \) is arbitrary.

11. Problem 2.25 on p. 58.